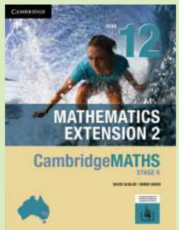
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

Topic summary and exercises:

With references to



(X2) Complex numbers

Name: ....

Initial version by H. Lam, August 2012. With major changes in October 2019 by I. Ham, and additional contributions from M. Ho in October 2022. Updated October 24, 2023 for latest syllabus.
Various corrections by students & members of the Mathematics Departments at North Sydney Boys High School and Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under (cc) CC BY 2.0.

Symbols used

- Beware! Heed warning.
- Provided on NESA Reference Sheet
- Facts/formulae to memorise.
- Literacy: note new word/phrase.
- Further reading/exercises to enrich your understanding and application of this topic.
- Syllabus specified content
- Facts/formulae to understand, as opposed to blatant memorisation.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\mathbb{C}$ the set of complex numbers

$\forall$ for all

Syllabus outcomes addressed

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

Syllabus subtopics

MEX-N1 Introduction to Complex Numbers

MEX-N2 Using Complex Numbers

Gentle reminder

- For a thorough understanding of the topic, *every* blank space/example question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Extension 2* (Sadler & Ward, 2019) and other selected texts will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

A new number system

1.1 Review of number systems

- ......Natural...... numbers. $\mathbb{N} = \{1, 2, 3 \cdots\}$

Example 1

Solve $x + 1 = 5$ and $x + 3 = 0$ over $\mathbb{N}$. **Answer:** $x = 4$, no solution

- ......Integers....... $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$

Example 2

Solve $x + 3 = 0$ and $2x + 4 = 7$ over $\mathbb{Z}$. **Answer:** $x = -3$, no solution

- ......Rational...... numbers. $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$

Example 3

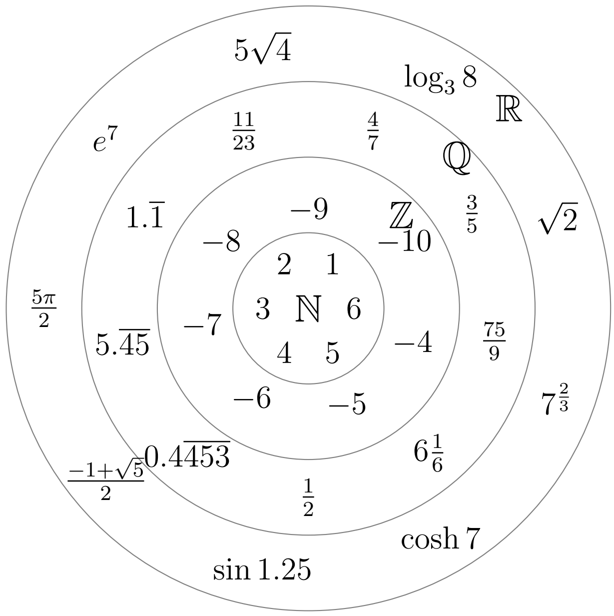
Solve $2x + 4 = 7$ and $x^2 - 2 = 0$ over $\mathbb{Q}$. **Answer:** $x = \frac{3}{2}$, no solution

- ...Real... numbers. $\mathbb{R}$

Example 4

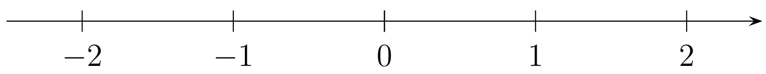
Solve $x^2 - 2 = 0$ and $x^2 + 5 = 0$ over $\mathbb{R}$. **Answer:** $x = \pm\sqrt{2}$, no solution

- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$



1.2 Rotation

- From $x = 1$, go to $x = -1$ by rotating π radians in the usual direction.
  - Multiply 1 by -1 to obtain -1 corresponds to rotating by π radians.
- Stop halfway whilst rotating? Quarter of way whilst rotating?



1.3 The "imaginary" numbers

Definition 1

Imaginary number The imaginary number i to be the "quantity" to multiply with a real number when rotating anti-clockwise by $\frac{\pi}{2}$ about $x = 0$.

- "Jump off" the real number line.

Definition 2

The imaginary number i has property such that

$$i \times i = i^2 = -1$$

- Why?

Definition 3

The set of all imaginary numbers, called the ....complex numbers........, is defined to be

$$\mathbb{C} = \{z : z = x + iy; x, y \in \mathbb{R}\}$$

Example 5

Find the values of i^2, i^3, i^4 and i^5.

- $i^2 =$ ....
- $i^3 =$ ....
- $i^4 =$ ....
- $i^5 =$ ....

Definition 4

Complex number A complex number z has ....real.... and ....imaginary.... parts and is defined by $z = x + iy$.

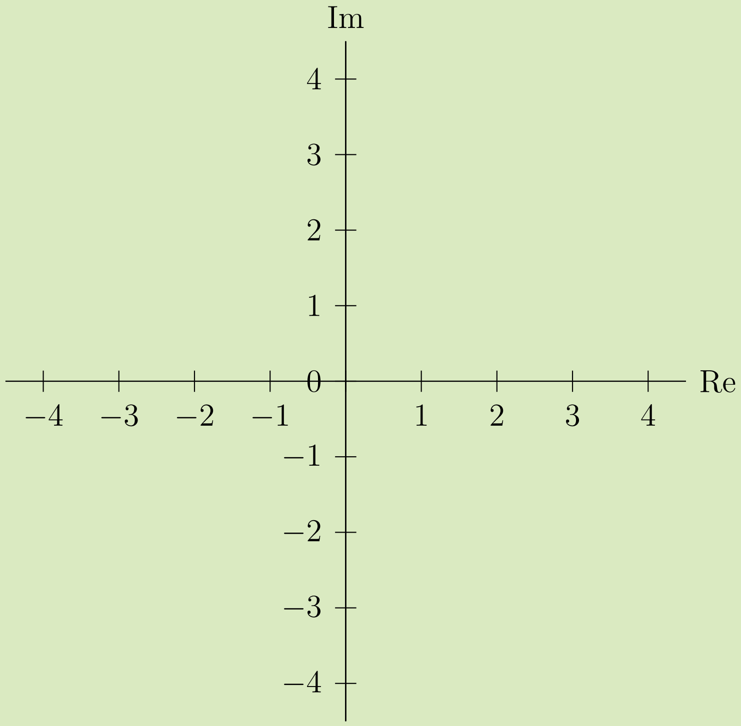
- The ....real.... part of z: $\text{Re}(z) = x$.
- The ....imaginary.... part of z: $\text{Im}(z) = y$.

- Treat real and imaginary parts as ....components.... of a complex number.
- $z = x + iy$ is known as ....Cartesian.... form.
- Plot on ....Argand.... ....diagram...., similar to plotting points coordinate geometry.

Example 6

On the following diagram, plot the location of:

- $z_1 = 3 + 4i$.
- $z_2 = 2 - i$.
- $z_3 = -1 - 3i$.
- $z_4 = -\frac{1}{2} + \frac{3}{2}i$.



Important note

Looks like another familiar topic from the Extension 1 course?

1.4 Basic operations with complex numbers

Example 7

Find the value of

1. $(2+\sqrt{3})-(5-4\sqrt{3})$ **2.** $(2+\sqrt{3})(5-4\sqrt{3})$

1.4.1 Addition

- Operations similar to surds (group rational parts with rational parts, irrational parts with irrational parts).
- Group ......real...... parts with ......real...... parts
- Group ....imaginary.......... parts with ....imaginary.......... parts.

1.4.2 Multiplication

- Use distributive law.
- Beware that $i^2 =$ ...-1..., which becomes ...real...

Example 8

If $z_1 = 2+3i$ and $z_2 = -1+5i$, find the value of

(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $3z_1$ (d) $3iz_1$ (e) z_1z_2

Example 9

Find $z \in \mathbb{C}$ such that $\text{Re}(z) = 2$ and z^2 is imaginary.

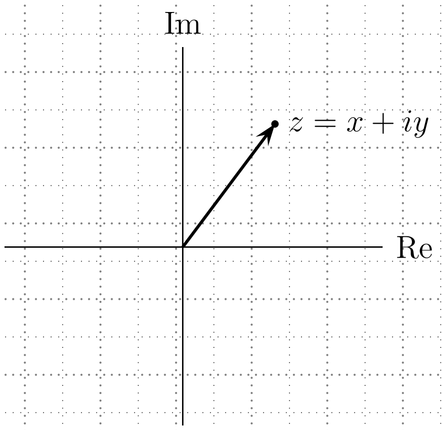
1.4.3 Complex conjugate pairs

Definition 5

If $z = x + iy$, then its complex conjugate is denoted $\overline{z}$ such that

$$\overline{z} = x - iy$$

- Analogous to conjugate surds, where the conjugate of $a + b\sqrt{c}$ is $a - b\sqrt{c}$
- Geometrically,



Example 10

If $z_1 = 2 + i$ and $z_2 = 1 - 3i$, evaluate in Cartesian form:

(a) $z_1 + z_2$

(b) $z_2 - z_1$

(c) $-z_2$

(d) $\overline{z_1}$

(e) $z_1 - \overline{z_2}$

(f) $\dfrac{1}{z_2}$

Laws/Results

Summary of complex number properties These involve the Cartesian form:

1. $\overline{z_1} + \overline{z_2} =$ $\overline{z_1 + z_2}$ . . . .

2. $\overline{z_1 z_2} =$ $\overline{z_1}\,\overline{z_2}$ . . . .

3. $z + \overline{z} =$ $\big(2\,\mathrm{Re}(z)\big)$ . . . .

4. $z - \overline{z} =$ $\big(2i\,\mathrm{Im}(z)\big)$ . . . .

Proof

1. Let $z_1 = x + iy$ and $z_2 = a + ib$

2. Let $z_1 = x + iy$ and $z_2 = a + ib$

3. Let $z = x + iy$

4. Let $z = x + iy$

History

Gerolamo Cardano (1501-1576), Mathematician (gambler and chess player!), published solutions to the cubic $ax^3 + bx + c = 0$ in *Ars Magna*. Cardano was one of the first to acknowledge the existence of imaginary numbers. Given during the Renaissance, negative numbers were treated suspiciously, imaginary numbers would have been almost heretical.

Cardano did not avoid (as most contemporaries did) nor did he immediately provide solutions to these imaginary numbers (possibly 200 years away). With the equations containing complex conjugate pairs, Cardano multiplied them together and obtained real numbers:

> Putting aside the mental tortures involved, multiply $5 + \sqrt{-15}$ with $5 - \sqrt{-15}$, making $25 - (-15)$, which is -15. Hence the product is 40.

Cardano, remarked in another work, that $\sqrt{-9}$ is neither $+3$ or -3, but some "obscure sort of thing".

Source:

- Wikipedia
  (`http://en.wikipedia.org/wiki/Gerolamo_Cardano`)
- *Complex and unpredictable Cardano*, Artur Ekert, Mathematical Institute, University of Oxford, United Kingdom
  (`http://www.arturekert.org/Site/Varia_files/NewCardano.pdf`)

Further exercises

Ex 1A (Sadler & Ward, 2019)

- All questions

Other references

- Lee (2006, Ex 2.3)

1.5 Properties of complex numbers

1.5.1 Equality

Theorem 1

Two complex numbers z_1 and z_2 are equal iff the real and imaginary parts are equal.

For $z_1 = a + ib$, $z_2 = c + id$, if $z_1 = z_2$, then

$$a + i\,b = c + i\,d$$

$$(a - c) + i\,(b - d) = 0 + 0i$$

Proof If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal and $z_1, z_2 \in \mathbb{C}$, then $x_1 = x_2$ and $y_1 = y_2$.

Steps

1. Assume $y_1 \neq y_2$:
2. $\therefore y_1 = y_2$ by contradiction.
3. Hence $x_1 = x_2$ and $y_1 = y_2$.

1.5.2 Solutions to equations

Example 11

Solve $z^2 + 1 = 0$ for $z \in \mathbb{C}$. **Answer:** $z = \pm i$

Example 12

Solve $z^2 + 2z + 10 = 0$ for $z \in \mathbb{C}$. **Answer:** $z = -1 \pm 3i$

Example 13

Solve $2z^2 + (1-i)z + (1-i) = 0$ for $z \in \mathbb{C}$. **Answer:** $z = i$, $z = -\frac{1}{2} - \frac{1}{2}i$

Example 14

Find the square roots of $-3+4i$ in Cartesian form.

Fill in the spaces

Observations

- Equations with ....real.... coefficients will have ....complex........ ....conjugate........ roots.
- Equations with ....complex........ coefficients do not necessarily have ....complex........ ....conjugate........ roots.

Further exercises

Ex 1B (Sadler & Ward, 2019)

- All questions

Other sources

- Fitzpatrick (1991, Ex 31(a), (b), (c))
- Lee (2006, Ex 2.1, 2.2)
- Arnold and Arnold (2000, Ex 2.1)

Section 2

Further arithmetic & algebra of complex numbers

2.1 The Argand diagram

Definition 6

The Argand diagram (or *complex number plane*) is a plane equivalent to the ......Cartesian...... plane, for displaying complex numbers. Each complex number $z = x + iy$ corresponds to a point ...$Z(x, y)$... on the Cartesian plane.

Fill in the spaces

- **Real component** is plotted on the ......horizontal...... axis.
- **Imaginary component** is plotted on the ......vertical...... axis.

Laws/Results

Equal complex numbers represent the same ...point... on the Argand diagram.

Important note

This theorem will be further explored in Section 4.

Theorem 2

Polynomials of complex numbers with real coefficients have ....complex........ ....conjugate........ roots.

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \qquad \text{where } a_0, a_1, \cdots a_n \in \mathbb{R}$$

If z_0 is a complex root of $P(z)$, then $\overline{z_0}$ is also a complex root of $P(z)$.

Proof If z_0 is a zero of $P(z)$, then $\overline{z_0}$ is also a zero, given $P(z)$ has real coefficients.

Steps

1. If z_0 be a zero of $P(z)$, then $P(z_0) = 0$.

2. Show that $P(\overline{z_0}) = \overline{P(z_0)}$.
(*Hint:* $\overline{z_1} + \overline{z_2} =$ ....$\overline{z_1 + z_2}$.... *and* $\overline{z_1} \cdot \overline{z_2} =$ ....$\overline{z_1 z_2}$....)

3. Hence $P(\overline{z_0}) = 0$ and $\overline{z_0}$ is a ....zero.... of $P(z)$.

2.2 Modulus/argument of a complex number

2.2.1 Natural ordering

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are *well ordered*:

  – $6 > 1$ – $\frac{22}{7} > \pi$ – $\frac{5}{4} > \frac{5}{7}$

- However:

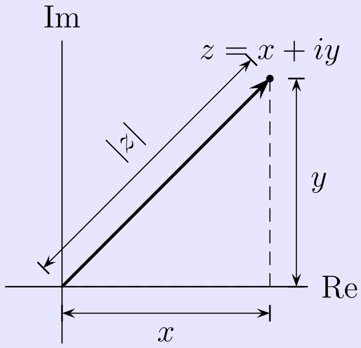
  – $6 + 4i$??? $3 + 2i$ – $3 - 3i$??? $2 + i$

- Natural ordering does not exist with complex numbers.

2.2.2 Modulus

Definition 7

The **modulus** of a complex number, denoted $|z|$ (where $z = x + iy$) is the magnitude (length) of the vector from O to z on the Argand diagram.



i.e.

$$|z| = \sqrt{x^2 + y^2}$$

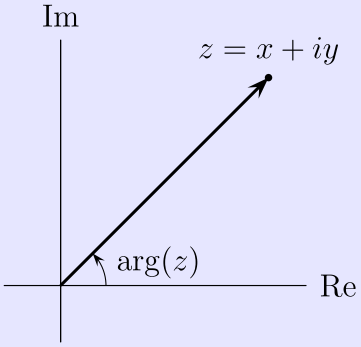
2.2.3 (Principal) Argument

Definition 8

The **argument** of a complex number, measured in ......radians...... , is denoted

$$\arg(z)$$

(where $z = x + iy$) is the ....angle.... that the vector from O to z makes with the positive real axis on the Argand diagram, with angles increasing in the anticlockwise direction.



i.e.

$$\arg(z) = \tan^{-1}\frac{y}{x}$$

- Duplicate argument(s)?

Example 15

Evaluate $\arg(z)$, where $z = 1 + i$.

Definition 9

The **principal argument** of a complex number, denoted

$$\mathrm{Arg}(z)$$

lies within the domain $-\pi < \mathrm{Arg}(z) \leq \pi$.

Important note

- The principal argument is generally quoted henceforth.
- Be aware of the quadrant which z lies. Inputting $\tan^{-1}\frac{y}{x}$ on the calculator mindlessly may give an erroneous result.
- The complex number $z = 0 + 0i$ has ..no.. argument ....defined.... .

Example 16

Find the modulus and principal argument of the following:

(a) $2 + 2i$ **Answer:** modulus: $2\sqrt{2}$, argument $\frac{\pi}{4}$

(b) $-1 - i\sqrt{3}$ **Answer:** modulus: 2, argument $-\frac{2\pi}{3}$

Example 17

[2011 HSC Q2] Let $w = 2 - 3i$ and $z = 3 + 4i$.

(a) Find $w + z$. **Answer:** $5 + i$

(b) Find $|w|$. **Answer:** $\sqrt{13}$

(c) Express $\dfrac{w}{z}$ in the form $a + ib$, where $a, b \in \mathbb{R}$. **Answer:** $-\frac{6}{25} - \frac{17}{25}i$

Further exercises

Ex 1C

- All questions

Ex 1D

- All questions

2.2.4 Triangle inequality

Theorem 3

For every complex number z_1 and z_2,

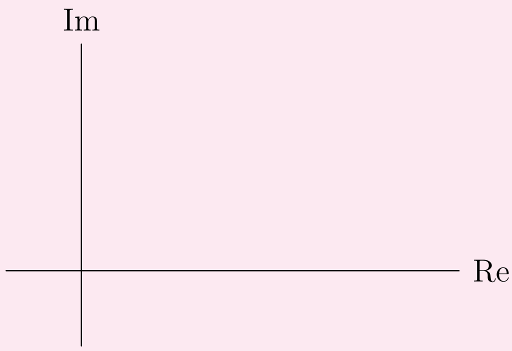
$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof [1]

Steps

1. Let $\mathbf{p}$ and $\mathbf{q}$ (with P and Q being the head of the arrow) represent the complex numbers z_1 and z_2 respectively, $\mathbf{p} + \mathbf{q}$ with R being the head of the arrow.

2. On the Argand diagram:



3. $|z_1 + z_2| = |z_1| + |z_2|$ iff O, P and Q are collinear (which implies $OP \parallel OQ \parallel OR$)

- Conclusion: $z_1 = kz_2$, where $k \in \mathbb{R}$ as vectors are parallel.

4. Otherwise, $|z_1 + z_2| < |z_1| + |z_2|$.

5. Hence, $|z_1 + z_2| \leq |z_1| + |z_2|$

[1] Never attempt to prove this algebraically!

Example 18

If $z_1 = 3 + 4i$ and $|z_2| = 13$, find the greatest value of $|z_1 + z_2|$. If $|z_1 + z_2|$ is at its greatest value, find the value of z_2 in Cartesian form.

Answer: $|z_1 + z_2| = 18$ at its greatest; $z_2 = \frac{39}{5} + \frac{52}{5}i$

Further exercises

Other references

- Lee (2006, Ex 2.6 Q1-7)
- Patel (2004, Ex 4K)
- Arnold and Arnold (2000, Ex 2.3)

2.3 Vector representation

2.3.1 Equivalence

Definition 10

Two vectors **p** and **q** on the Argand diagram are **equal** iff *both*

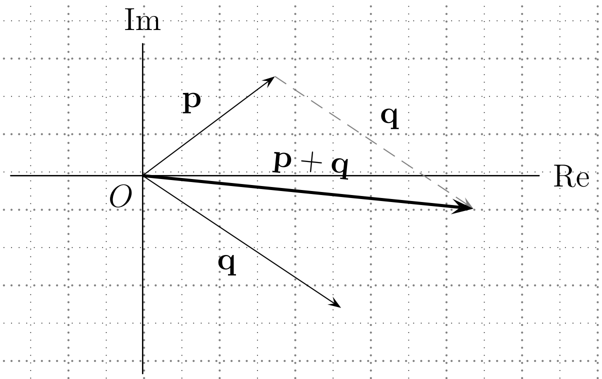
- Modulus (magnitude), and
- Argument (direction)

are equal.

- The starting point (tail) is irrelevant for a vector.

2.3.2 Addition

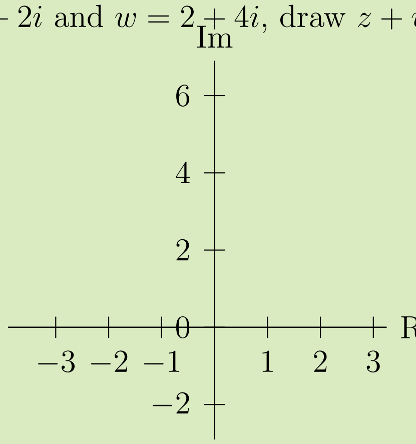
- Place vectors, head-to-tail.

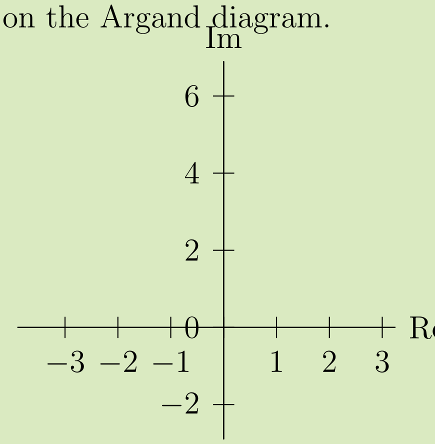


- Parallelogram of vectors when adding two vectors.

Example 19

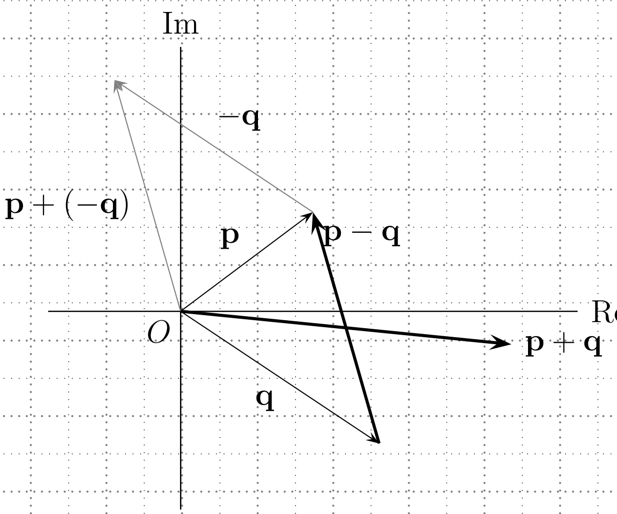
If $z = -3 + 2i$ and $w = 2 + 4i$, draw $z + w$ on the Argand diagram.





2.3.3 Subtraction

- For $z_1 - z_2$, add $-z_2$ to z_1.

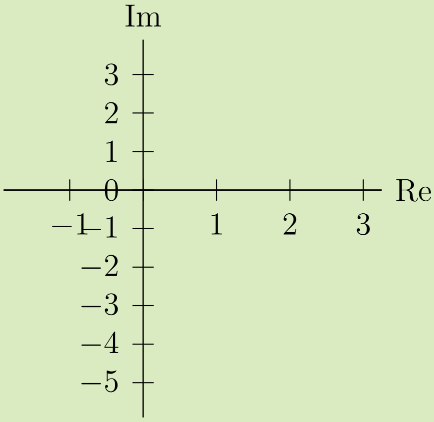


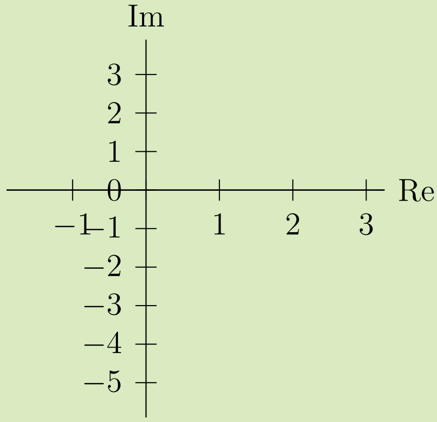
- Alternatively, "what vector get you to z_1 from z_2?"

- Quick & easy parallelogram of vectors:
  1. $z_1 + z_2$ starts from tails of z_1 & z_2,
  2. $z_2 - z_1$ starts from head of z_1, goes to head of z_2.

Example 20

If $z = 3 - 2i$ and $w = 2 - 5i$, draw $z - w$ on the Argand diagram



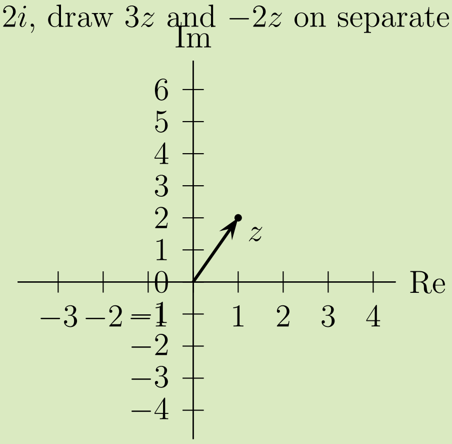


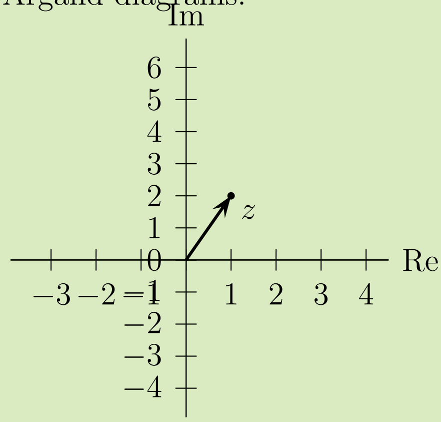
2.3.4 Scalar multiplication

- For kz_1 where $k \in \mathbb{R}$, stretch z_1 by factor of k.
- If $k < 0$, direction of new vector is opposite to original vector.

Example 21

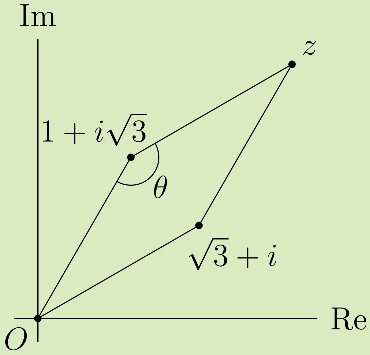
If $z = 1 + 2i$, draw $3z$ and $-2z$ on separate Argand diagrams.





Example 22

[2011 HSC Q2] On the Argand diagram, the complex numbers 0, $1+i\sqrt{3}$, $\sqrt{3}+i$, and z form a rhombus.



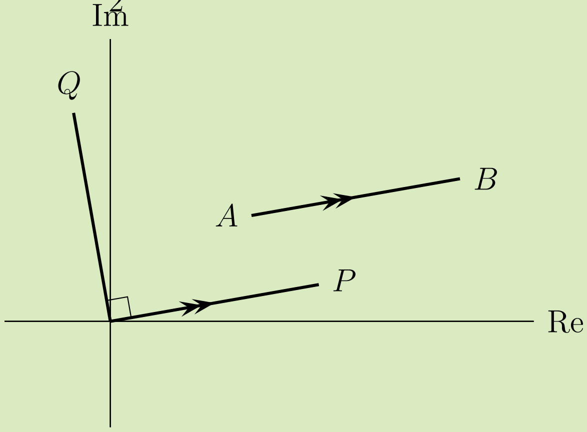
(i) Find z in the form $a+ib$, where a and b are real numbers.

(ii) An interior angle, θ, of the rhombus is marked on the diagram. Find the value of θ.

Answer: $z = (\sqrt{3}+1)+i(\sqrt{3}+1)$, $\theta = \frac{5\pi}{6}$

Example 23

In the Argand diagram below, intervals AB, OP and OQ are equal in length, OP is parallel to AB and $\angle POQ = \dfrac{\pi}{2}$.



(a) If A and B represent the complex numbers $3 + 5i$ and $9 + 8i$ respectively, find the complex number which is represented by P.

(b) Hence find the complex number which is represented by Q.

Example 24

(Sadler & Ward, 2019) Let $z = 1 + i$.

(a) Find, in Cartesian form, the complex number w such that wz is a rotation of z by $\frac{\pi}{3}$ anticlockwise about the origin.

(b) Evaluate wz in Cartesian form.

(c) Verify $|wz| = |z|$, then plot z and wz on an Argand diagram.

Further exercises

Ex 1E

- Q1-21

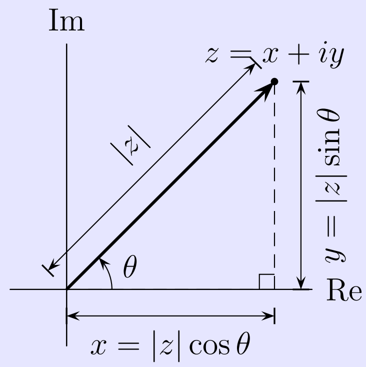
2.4 Polar form (Euler's formula)

2.4.1 Arithmetic

Definition 11

The **modulus-argument form** of a complex number z is

$$\begin{aligned} z &= x + iy && \text{(Cartesian form)} \\ &= |z|\cos\theta + i\,|z|\sin\theta \\ &= |z|\,(\cos\theta + i\sin\theta) \\ &= r\,(\cos\theta + i\sin\theta) && \text{(Mod-arg form)} \end{aligned}$$



where $\text{Arg}(z) = \theta$.

- $z = r(\cos\theta + i\sin\theta)$ often abbreviated[2] to $z = r\,\text{cis}\,\theta$.
- Better to abbreviate $z = r(\cos\theta + i\sin\theta)$ to $z = re^{i\theta}$ (for reasons that will be made obvious later)

Definition 12

Polar form: Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $e \approx 2.71828\cdots$

(Leonhard Euler, 1707-1783. `http://en.wikipedia.org/wiki/Leonhard_Euler`)

(All index laws in $\mathbb{R}$ also apply to the complex exponential).

[2] $\text{cis}\,\theta$ does very little to assist your understanding of the rules for multiplying complex numbers!

Example 25

Write $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ in Cartesian form. **Answer:** $z = -1 + i$

Example 26

Write $z = -2 - 2i$ in polar form. **Answer:** $z = 2\sqrt{2}e^{-i\frac{3\pi}{4}}$

Example 27

Write $z = \dfrac{10}{\sqrt{3} + i}$ in polar form, and hence write in simplest Cartesian form.

Answer: $z = 5e^{-\frac{i\pi}{6}} = \frac{5\sqrt{3}}{2} - \frac{5}{2}i$

Example 28

Write $z = 6e^{-\frac{2i\pi}{3}}$ in Cartesian form. **Answer:** $-3 - 3i\sqrt{3}$

Example 29

Evaluate the product $(1+i)\left(1-i\sqrt{3}\right)$ in Cartesian form and polar form, to show that $\cos\dfrac{\pi}{12}=\dfrac{1+\sqrt{3}}{2\sqrt{2}}$.

Example 30

Use Euler's formula to write $\cos\theta$ and $\sin\theta$ in terms of e.

Further exercises

Ex 3D

- Q1-15

Other references

- Patel (2004, Ex 4C, Q1-10)
- Arnold and Arnold (2000, Ex 2.2, Q1-4)
- Lee (2006, Ex 2.6 Q8 onwards)

2.4.2 Multiplication/Division

Laws/Results

Multiplication of z_1 and z_2: moduli ....multiply...., arguments ..add..

$$z_1 z_2 = r_1 r_2 \left(\cos\left(\theta_1 + \theta_2\right) + i \sin\left(\theta_1 + \theta_2\right)\right) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Division of z_1 and z_2: moduli ....divide...., arguments ....subtract....

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left(\cos\left(\theta_1 - \theta_2\right) + i \sin\left(\theta_1 - \theta_2\right)\right) = r_1 r_2 e^{i(\theta_1 - \theta_2)}$$

Proof Let $z_1 = r_1 (\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2 (\cos\theta_2 + i \sin\theta_2)$

$$z_1 z_2 =$$

Proof (via index laws and complex exponential) Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

$$z_1 z_2 =$$

2.4.3 Rotations/Enlargement

Laws/Results

Multiplication of z by another complex number $\omega = re^{i\theta} = r(\cos\theta + i\sin\theta)$:

- Enlarges the modulus of z by factor r
- Rotates z anticlockwise about the origin by θ.
- (Multiplication by $\omega = re^{-i\theta}$) : Rotates z clockwise about the origin by θ.

Example 31

Let $z = \sqrt{3} + i$. Multiply z by another complex number $\omega = e^{i\frac{\pi}{2}}$. Find $z\omega$ in Cartesian form. Plot z and iz on an Argand diagram.

Laws/Results

Multiplication of z by i and $-i$ respectively:

- Rotates z anticlockwise about the origin by $\frac{\pi}{2}$
- Rotates z clockwise about the origin by $\frac{\pi}{2}$

(Essentially, multiply by $e^{\pm i\frac{\pi}{2}}$)

Example 32

[UNSW MATH1131 exercises, Problems 1.7, Q35]

(a) Explain why multiplying a complex number z by $e^{i\theta}$ rotates the point represented by z anticlockwise about the origin, through an angle θ.

(b) The point represented by the complex number $1+i$ is rotated anticlockwise about the origin through an angle of $\dfrac{\pi}{6}$. Find the resultant complex number in polar and Cartesian form.

(c) Find the complex number (in Cartesian form) obtained by rotating $6-7i$ anticlockwise about the origin through an angle $\dfrac{3\pi}{4}$.

Answer: (a) Explain (b) $\sqrt{2}e^{i\frac{5\pi}{12}} = \frac{1}{2}\left(\left(\sqrt{3}-1\right)+i\left(\sqrt{3}+1\right)\right)$ (c) $\frac{1}{\sqrt{2}}(1+13i)$

2.4.4 Conjugates

Laws/Results

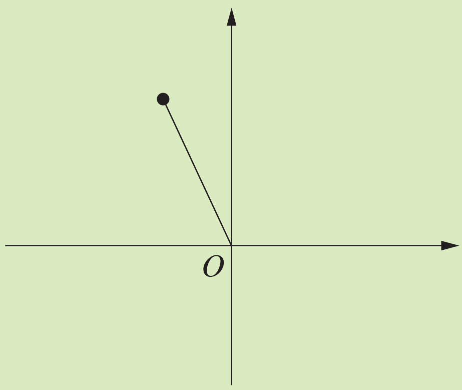
If $z = r(\cos\theta + i\sin\theta)$, then

$$\overline{z} = r(\cos(-\theta) + i\sin(-\theta)) = re^{-i\theta}$$

Proof Let $z = r(\cos\theta + i\sin\theta)$.

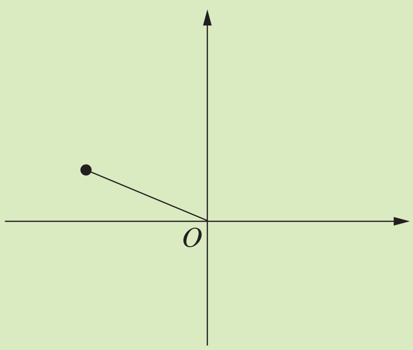
Example 33

[2020 Ext 2 Sample Q4] The Argand diagram shows the complex number $e^{i\theta}$.

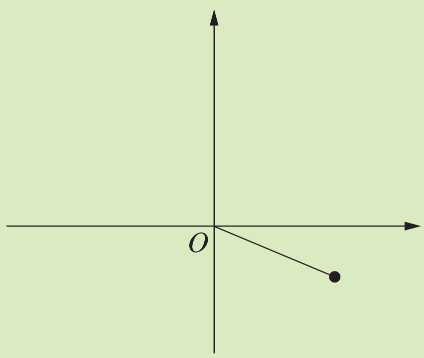


Which of the following could be the complex number $-ie^{-i\theta}$?

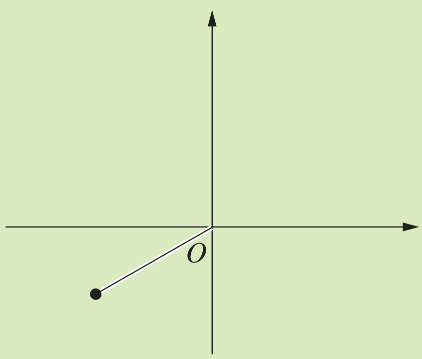
(A)



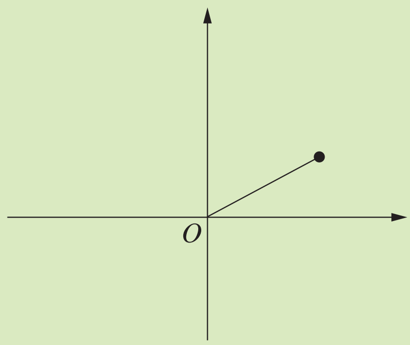
(C)



(B)

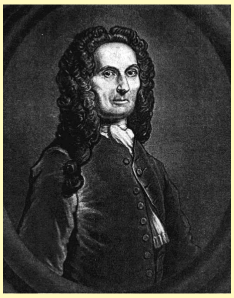


(D)



2.4.5 Powers

Theorem 4



De Moivre's Theorem For $n \in \mathbb{Z}$,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

(Abraham De Moivre, 1667–1754.
`http://en.wikipedia.org/wiki/Abraham_de_Moivre`)

Proof (via complex exponential)

Proof (by induction – for later in the course)

Example 34

Simplify the following, expressing the answer in polar form:

(a) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3$. **Answer:** -1

(b) $\left[2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^3$. **Answer:** $8e^{i\frac{\pi}{4}}$

(c) $\left[\sqrt{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\right]^8$. **Answer:** $16e^{i\frac{2\pi}{3}}$

(d) $\left[\frac{1}{2}\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)\right]^4 \div \left[\frac{1}{3}\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)\right]^5$ **Answer:** $\frac{243}{16}e^{-i\frac{39\pi}{40}}$

Example 35

If $|z_1| = 3$, $\text{Arg}(z_1) = 2$, $|z_2| = 2$ and $\text{Arg}(z_2) = 3$, find the modulus and argument of $\dfrac{2{z_1}^2}{5{z_2}^3}$.

Answer: $|z| = \frac{9}{20}$, $\text{Arg}(z) = 2\pi - 5$

Example 36

[1988 4U HSC Q4(a)]

(a) Express $z = \sqrt{2} - i\sqrt{2}$ in modulus-argument form.

Answer: $z = 2\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

(b) Hence write z^{22} in the form $a + ib$, $a, b \in \mathbb{R}$.

Answer: $z^{22} = 2^{22}i$

Example 37

(a) If $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$, find the moduli and principal arguments of z_1, z_2 and $\dfrac{z_1}{z_2}$.

Answer: $z_1 = \sqrt{2}\exp\left(\frac{i\pi}{4}\right)$, $z_2 = 2\exp\left(-\frac{i\pi}{6}\right)$, $\frac{z_1}{z_2} = \frac{1}{\sqrt{2}}\exp\left(\frac{5i\pi}{12}\right)$.

(b) If $z = \dfrac{1+i}{\sqrt{3}-i}$, find the smallest positive integer n such that z^n is real, and evaluate z^n for this integer n.

Answer: $n = 12$, $z^{12} = -\frac{1}{64}$

Laws/Results

Summary of complex number properties These involve the modulus-argument form:

1. $\overline{z_1} + \overline{z_2} =$ $\overline{z_1 + z_2}$
2. $\overline{z_1 z_2} =$ $\overline{z_1}\,\overline{z_2}$
3. $z + \overline{z} =$ $\left(2\,\mathrm{Re}(z)\right)$
4. $z - \overline{z} =$ $\left(2i\,\mathrm{Im}(z)\right)$
5. $|z_1 z_2| =$ $|z_1|\,|z_2|$, $\arg(z_1 z_2) =$ $\arg z_1 + \arg z_2$
6. $\left|\dfrac{z_1}{z_2}\right| =$ $\dfrac{|z_1|}{|z_2|}$, $\arg\left(\dfrac{z_1}{z_2}\right) =$ $\arg(z_1) - \arg(z_2)$
7. $|z^n| =$ $|z|^n$, $\arg(z^n) =$ $n\arg(z)$
8. $\left|\dfrac{1}{z^n}\right| =$ $\dfrac{1}{|z|^n}$, $\arg\left(\dfrac{1}{z^n}\right) =$ $-n\arg(z)$

Proof

1. Let $z_1 = r_1(\cos\theta + i\sin\theta)$ and $z_2 = r_2(\cos\phi + i\sin\phi)$

2. Let $z_1 = r_1(\cos\theta + i\sin\theta)$ and $z_2 = r_2(\cos\phi + i\sin\phi)$

3. Let $z = r(\cos\theta + i\sin\theta)$

4. Let $z = r(\cos\theta + i\sin\theta)$

5. Let $z_1 = r_1(\cos\theta + i\sin\theta)$ and $z_2 = r_2(\cos\phi + i\sin\phi)$

6. Let $z_1 = r_1(\cos\theta + i\sin\theta)$ and $z_2 = r_2(\cos\phi + i\sin\phi)$

7. Let $z = r(\cos\theta + i\sin\theta)$

8. Let $z = r(\cos\theta + i\sin\theta)$

Further exercises

Note all uses of 'cis θ' should really be replaced with $e^{i\theta}$.

Ex 3A

- Q1-17

Other references

- Patel (2004, Ex 4C, Q11 onwards),
- Patel (2004, Ex 4D)
- Arnold and Arnold (2000, Ex 2.2, Q6 onwards)
- Lee (2006, Ex 2.5, 2.9)

Section 3

Curves and regions in the complex plane

3.1 Curves

3.1.1 Lines/rays

- $|z| = r$

Derivation of Cartesian equation:

Diagram:

Description: ......Circle......, ...centre...... $(0, 0)$, ...radius...... r

Write the equation that represents the ......circle...... with ......centre...... z_1 and ......radius...... r:

- $|z - z_1| = |z - z_2|$

  Diagram:

  Description: Perpendicular bisector of the interval joining z_1 to z_2.

- $\text{Arg}(z - z_1) = \alpha,\ \alpha \in \mathbb{R}$

  Diagram:

  Description: ....

- $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$

  Diagram:

  Description: Line through z_1 and z_2 but excluding the segment from z_1 to z_2

3.1.2 Curves

- $|z - z_1| = r$

  Derivation of Cartesian equation: **Diagram:**

  Description: Circle, centred at z_1, radius r

- ⚠ $\text{Arg}(z - z_1) - \text{Arg}(z - z_2) = \alpha$, $0 < \alpha < \pi$

  Diagrams:

  Origin: circle geometry theorem – *Angle at the circumference subtended by the same arc/chord*

  Description: ....

- ⚠ $2\,\text{Arg}(z - z_1) = \text{Arg}(z - z_2) = \alpha$, $0 < \alpha < \frac{\pi}{2}$

  Diagram:

  Origin: circle geometry theorem – *Angle at the centre is double the angle at the circumference subtended by the same arc/chord*

  Description: ....

Example 38

For the following:

i. Describe the path traced out by the conditions on z.

ii. Draw a sketch.

iii. Give the Cartesian equation of the line/curve.

(a) $|z| = 2$

(b) $z\overline{z} = 16$

(c) $|z+2| = 1$

(d) $|z+2+3i| = 2$

Answer: (a) $x^2+y^2=4$ (b) $x^2+y^2=16$ (c) $(x+2)^2+y^2=1$ (d) $(x+2)^2+(y+3)^2=4$

Example 39

[2019 NBHS Ext 2 Trial Q11]

i. Find the points of intersection on the curves given by **3**

$$|z - i| = 1 \text{ and } \operatorname{Re}(z) = -\frac{1}{\sqrt{3}}\operatorname{Im}(z)$$

ii. Sketch above the two curves on the Argand diagram to show the points of intersection. **1**

Answer: $0 + 0i,\ -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

Example 40

[2018 Independent Ext 2 Q12] z is a complex number such that

$$\left|z - 2\sqrt{2}\,(1+i)\right| = 2$$

i. On an Argand diagram, sketch the path which is traced by the condition above. **1**

ii. Q is a point on the path traced out where z has its smallest principal argument. **2**

Find the value of the complex number represented by Q in modulus-argument form.

Answer: $2\sqrt{3}e^{i\frac{\pi}{12}}$

Example 41

[2012 NSGHS Ext 2 Q12] Given z is a complex number, sketch on the number plane, the path traced out by the complex numbers z such that

$$\arg z = \arg\Big(z - (1+i)\Big)$$

Example 42

Sketch the curve in the Argand diagram determined by $\text{Arg}(z-1) = \text{Arg}(z+1) + \frac{\pi}{4}$. Find its Cartesian equation.

Answer: $x^2 + (y-1)^2 = 2,\ y > 0$

Example 43

[2016 Caringbah HS Ext 2, Q8] ⚠ The complex number z satisfies

$$\text{Arg}\left(\frac{z-2}{z+2i}\right) = -\frac{\pi}{2}$$

Find the maximum value of $|z|$.

(A) $\sqrt{2}$ (B) $2\sqrt{2}$ (C) $2-\sqrt{2}$ (D) $2+\sqrt{2}$

Example 44

z satisfies $|z-i| = \text{Im}(z)+1$. Sketch the path traced out by the point P representing z in the Argand diagram and write down its Cartesian equation. **Answer:** $x^2 = 4y$

Example 45

[2003 Q2] Suppose that the complex number z lies on the unit circle, and $0 \leq \arg(z) \leq \frac{\pi}{2}$.

Prove that $2\arg(z+1) = \arg(z)$.

⚠ Draw picture!

3.2 Regions

For a brief review Stage 5 (Year 10) work on regions:

Further exercises

Ex 3F (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019)

- All questions

Example 46

[2011 CSSA Ext 2 Q2]

i. Sketch the path traced out by the complex numbers z such that **2**

$$|z - 3 + 3i| \leq 2$$

ii. Find the maximum value of $|z|$. **1**

Example 47

Sketch the region in the Argand diagram defined simultaneously by

$$6 \leq \operatorname{Re}\left[(2 - 3i)z\right] < 12 \quad \text{and} \quad \operatorname{Re}(z)\operatorname{Im}(z) > 0$$

Example 48

Draw a sketch of the curve or region, given $z \in \mathbb{C}$ and

(a) $\text{Arg}(z) = \frac{\pi}{3}$ (b) $0 \leq \text{Arg}(z) \leq \frac{2\pi}{3}$ (c) $\text{Arg}(z - 2 + 3i) = \frac{\pi}{4}$

Example 49

z is a complex number which simultaneously satisfies

$$2 \leq |z+3| \leq 3 \quad \text{and} \quad 0 \leq \text{Arg}(z+3) \leq \frac{\pi}{3}$$

Find the area and perimeter of the region in the Argand diagram determined by these restrictions on z. **Answer:** $A = \frac{5\pi}{6}$ units^2, $P = 2 + \frac{5\pi}{3}$ units

Further exercises

Ex 1F

- Q1-17

Other resources

- Patel (1990, Self Testing Ex 4.9, p.127)
- Arnold and Arnold (2000, Ex. 2.5)
- Fitzpatrick (1991, Ex 31(f))
- Lee (2006, Ex 2.7, 2.8)

Section 4

Applications to polynomials

4.1 Polynomials theorems for equations with roots in $\mathbb{C}$

Laws/Results

For polynomials with real coefficients, the following theorems function in $\mathbb{C}$, exactly in the same way as they do in $\mathbb{R}$.

- Remainder theorem.
  - Added bonus: conjugate roots means the conjugate remainder can be found easily.
- Factor theorem.
  - Added bonus: conjugate roots may help!
- Vieta's formulas :
  - Sum
  - Pairs
  - Triples etc
  - Product
- Roots with multiplicity > 1.

Example 50

(Sadler & Ward, 2019) Let $P(x) = x^3 - 2x^2 - x + k$, $k \in \mathbb{R}$.

(a) Show that $P(i) = (2 + k) - 2i$

(b) When $P(x)$ is divided by $x^2 + 1$, the remainder is $4 - 2x$. Find the value of k.

Answer: $k = 2$

Example 51

Find all the zeros of $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$ over $\mathbb{C}$, given $1 + i$ is a zero. Hence, fully factorise $P(x)$ over $\mathbb{R}$. **Answer:** $P(x) = (x^2 - 2x + 2)(x + 2)(x - 1)$

Example 52

Prove that $2 + i$ is a root of the equation $x^4 - 2x^3 - 7x^2 + 26x - 20 = 0$, and hence solve the equation completely over $\mathbb{C}$. **Answer:** $x = 2 \pm i,\ -1 \pm \sqrt{5}$.

Example 53

Determine b, $c \in \mathbb{R}$ such that $-2i$ is a zero of $x^3 + 3x^2 + bx + c$. **Answer:** $b = 4$, $c = 12$.

Example 54

$P(x)$ is a monic polynomial of degree 4 with integer coefficients and constant term 2. $P(x)$ has a zero i, and a rational zero. The sum of the zeros of $P(x)$ is a positive real number. Find $P(x)$ factorised into irreducible factors over $\mathbb{R}$.

Answer: $P(x) = (x^2 + 1)(x - 1)(x - 2)$

Further exercises

Ex 1G

- Q1-16, 18

4.2 Trigonometric identities

Example 55

(a) **[2003 Ext 2 HSC Q2(d)]** Using De Moivre's theorem, find an expression for $\cos 5\theta$ in terms of $\cos\theta$. **Answer:** $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(b) Hence solve $16x^4 - 20x^2 + 5 = 0$ for x. **Answer:** $x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$.

Solution

Steps

(a)
- By De Moivre's Theorem,

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

- Expand $(\cos\theta + i\sin\theta)^5$ via binomial theorem:

- Equate real parts & simplify:

(b)

Example 56

(a) Use De Moivre's Theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(b) Hence solve $8x^3 - 6x - 1 = 0$.

(c) Deduce that $\cos\dfrac{\pi}{9} = \cos\dfrac{2\pi}{9} + \cos\dfrac{4\pi}{9}$.

Answer: $x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$.

Example 57

(a) Use De Moivre's Theorem to express $\tan 5\theta$ in terms of powers of $\tan\theta$.

(b) Hence show that $x^4 - 10x^2 + 5 = 0$ has roots $\pm\tan\frac{\pi}{5}$ and $\pm\tan\frac{2\pi}{5}$.

(c) Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$.

(d) By solving $x^4 - 10x^2 + 5 = 0$ via another method, find the exact value of $\tan\frac{\pi}{5}$.

Answer: **(a)** $\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ **(b)** Show. **(c)** Show. **(d)** $\tan\frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$.

4.3 Further exercises

1. (a) Factorise $z^6 - 1$ into the real quadratic factors.

(b) Hence factorise $z^4 + z^2 + 1$.

2. (a) Show that the roots of $y^4 + y^3 + y^2 + y + 1 = 0$ are $y = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$, where $k = 1, 2, 3, 4$ and hence show that $\cos\frac{\pi}{5} = \frac{1}{2} + \cos\frac{2\pi}{5}$. Also prove that $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$.

(b) By letting $x = y + \frac{1}{y}$, show that the roots of $x^2 + x - 1 = 0$ are $2\cos\frac{2k\pi}{5}$, where $k = 1, 2$ and deduce that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$

(c) **(Method 2)**

Prove that $y^4 + y^3 + y^2 + y + 1 = \left(y^2 + 2y\cos\frac{\pi}{5} + 1\right)\left(y^2 - 2y\cos\frac{2\pi}{5} + 1\right)$ and hence deduce that $\cos\frac{\pi}{5} = \frac{1}{2} + \cos\frac{2\pi}{5}$

3. Suppose that $z^7 = 1$, $z \neq 1$

(a) Deduce that $z^3 + z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} = 0$

(b) By letting $x = z + \frac{1}{z}$, reduce the equation in (i) to a cubic equation in x.

(c) Hence deduce that $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$

4. (a) Express $\cos 3\theta$ in terms of $\cos\theta$

(b) Use the result to solve $8x^3 - 6x + 1 = 0$.

(c) Deduce that

i. $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$

ii. $\sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = 8$

5. (a) Express $\cos 3\theta$ and $\cos 2\theta$ in terms of $\cos\theta$

(b) Show that $\cos 3\theta = \cos 2\theta$ can be expressed as $4x^3 - 2x^2 - 3x + 1 = 0$, where $t = \tan\theta$

(c) By solving equation in (ii) for x, show that $\cos\dfrac{2\pi}{5} = \dfrac{\sqrt{5}-1}{4}$

6. (a) Factorise $z^6 + 1$ into real quadratic factors.

(b) Hence deduce that $\cos 3\theta = 4\left(\cos\theta - \cos\dfrac{\pi}{6}\right)\left(\cos\theta - \cos\dfrac{\pi}{2}\right)\left(\cos\theta - \cos\dfrac{5\pi}{6}\right)$

7. ⚠ Show that the roots of $(z-1)^6 + (z+1)^6 = 0$ are $\pm i$, $\pm i\cot\dfrac{\pi}{12}$, and $\pm i\cot\dfrac{5\pi}{12}$

Further exercises

Ex 3B

- Q1-4, 6-14

Other resources

- Lee (2006, Ex 2.11 (skip Q6(iii)))
- Patel (1990, Self Testing Ex 4.7 p.109)
- Arnold and Arnold (2000, Ex 2.4)

4.4 Roots of complex numbers

4.4.1 Factorisations of higher powers

Example 58

(a) Evaluate the following partial sum:

$$1 + r + r^2 + r^3 + r^4$$

(b) Hence factorise $x^5 - 1$ over $\mathbb{Z}$

Example 59

(a) Evaluate the following partial sum:

$$1 - r + r^2 - r^3 + r^4$$

(b) Hence factorise $x^5 + 1$ over $\mathbb{Z}$

Laws/Results

Difference of powers

$$x^n - y^n = (x - y)\ (x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$$

Sum of powers

$$x^n + y^n = (x + y)\ (x^{n-1} - x^{n-2}y + \cdots + y^{n-1})$$

- Signs alternate

4.4.2 Graphical solutions and consequent factorisations

- To find n-th roots of complex numbers, use De Moivre's Theorem and polar form.

Example 60

Find the cube roots of unity, i.e. solve $z^3 = 1$.

Solution (via De Moivre's Theorem)

Steps

1. $z^3 = 1\left(\cos 2k\pi + i \sin 2k\pi\right)$, where $k = \overbrace{0, 1, 2}^{\text{3 roots}}$.
2. Hence, $z = 1^{\frac{1}{3}}\left(\cos 2k\pi + i \sin 2k\pi\right)^{\frac{1}{3}} =$ $1^{\frac{1}{3}}\left(\cos \frac{2k\pi}{3} + \right)$ by De Moivre's Theorem.
3. Fix up "out of range" arguments (change to principal argument):

Solution (via polar form)

Example 61

[2011 HSC Q2] Find, in modulus-argument form, all solutions of $z^3 = 8$.

Answer: $z = 2, 2\left(\cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}\right)$

Example 62

Solve for z: $z^4 = -8 - 8\sqrt{3}i$, and plot the solutions on the Argand diagram.

Answer: $z = 2e^{i\frac{\pi}{3}}, 2e^{i\frac{5\pi}{6}}, 2e^{i\frac{-2\pi}{3}}, 2e^{-i\frac{\pi}{6}}$

Example 63

[2016 Ext 2 Q10] Suppose that $x + \dfrac{1}{x} = -1$.

What is the value of $x^{2016} + \dfrac{1}{x^{2016}}$?

(A) 1 (B) 2 (C) $\dfrac{2\pi}{3}$ (D) $\dfrac{4\pi}{3}$

Example 64

Find the fourth roots of $z = 1 + i\sqrt{3}$ in modulus-argument form.

Answer: $2^{\frac{1}{4}}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$, $2^{\frac{1}{4}}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$, $2^{\frac{1}{4}}\left(\cos\frac{11\pi}{12} - i\sin\frac{11\pi}{12}\right)$, $2^{\frac{1}{4}}\left(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}\right)$

Example 65

(a) Find the five fifth roots of unity and plot them on the unit circle.

(b) If ω is a non-real fifth root of unity, show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

(c) Hence or otherwise, factorise $z^5 - 1$ completely over $\mathbb{R}$.

Answer: $(z-1)\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$

Example 66

Find all the zeros of $P(x) = x^4 + x^3 + x^2 + x + 1$. Hence factorise $P(x)$ into irreducible factors over $\mathbb{R}$. Deduce that $\cos \dfrac{2\pi}{5} = \dfrac{\sqrt{5} - 1}{4}$. **Answer:** $x = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}, \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}$

Example 67

(a) Find the five fifth-roots of -1. **Answer:** $e^{i\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right)}$, where $k \in [0, 4]$

(b) If ω is a non-real fifth root of -1 with the smallest positive argument, show that $1 - \omega + \omega^2 - \omega^3 + \omega^4 = 0$.

(c) Find the exact values of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$.

Answer: $\cos\frac{\pi}{5} = \frac{1}{4}\left(1+\sqrt{5}\right)$, $\cos\frac{3\pi}{5} = \frac{1}{4}\left(1-\sqrt{5}\right)$

Example 68

[2014 JRAHS Trial Q15] Let α be a complex root of the polynomial $z^7 = 1$ with the smallest argument. Let $\theta = \alpha + \alpha^2 + \alpha^4$ and $\phi = \alpha^3 + \alpha^5 + \alpha^6$.

(i) Show that $\theta + \phi = -1$ and $\theta\phi = 2$. **3**

(ii) Write a quadratic equation whose roots are θ and ϕ. Hence show that **2**

$$\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2} \quad \text{and} \quad \phi = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$$

(iii) Show that **2**

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$$

4.4.3 Roots of unity: reduction from higher powers

Laws/Results

Change subject to highest power of x:

- If $x^2 + x + 1 = 0$, then

$$x^2 = -x - 1$$

- If $x^3 + x^2 + x + 1 = 0$, then

$$x^3 = -x^2 - x - 1$$

These results can be used creatively to reduce the powers down to more manageable powers.

Example 69

[Ex 3C Q1]

(a) Find the three cube roots of unity, expressing the complex roots in both modulus-argument form and Cartesian form.

(b) Show that the points in the complex plane representing these three roots form an equilateral triangle.

(c) If ω is one of the complex, non-real roots, show that the other complex root is ω^2.

(d) Write down the values of: i. ω^3 ii. $1 + \omega + \omega^2$

(e) Show that:

i. $(1 + \omega^2)^3 = -1$

ii. $(1 - \omega - \omega^2)(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 8$

iii. $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = 9$

Example 70

[2017 BHHS Ext 2 Trial Q1] If ω is an imaginary cube root of unity, then what is $(1+\omega-\omega^2)^{2017}$ equal to?

(A) $-2^{2017}\omega$ (B) $2^{2017}\omega$ (C) $-2^{2017}\omega^2$ (D) $2^{2017}\omega^2$

Example 71

[2016 JRAHS Ext 2 Trial HSC Q11] (3 marks) Simplify

$$\left(1+2\omega+3\omega^2\right)\left(1+3\omega+2\omega^2\right)$$

where ω is a complex cube root of unity.

Example 72

[2010 NSBHS Ext 2 Assessment Task 1] If w is a non-real cube root of unity, i.e. $w^3 = 1$,

i Prove $\dfrac{1}{1+w} + \dfrac{1}{1+w^2} = 1$ **3**

ii Show that **3**

$$\frac{1 + w^n + w^{2n}}{3} = \begin{cases} 1 & n \text{ is a multiple of } 3 \\ 0 & \text{otherwise} \end{cases}$$

Further exercises

Ex 3C

- Q1-11

Other resources

- Fitzpatrick (1991, Ex 31(a))
- Patel (2004, Ex 4D, 4E, 4I)
- Lee (2006, Ex 2.10)
- Arnold and Arnold (2000, Ex 2.4)

Section A

Past HSC questions

Important note

Whilst the legacy Extension 2 ('4 Unit') syllabus contained Complex Numbers, there have been several content sections that have now been removed for HSC examinations from 2020.

If in doubt, consult your teacher regarding whether a particular part is suitable to attempt or not.

Definition 13

Locus the path traced out by a point, subject to certain conditions.

This word was used extensively in the legacy syllabuses but has now been removed. Simply replace any instances of *locus* with *path traced out by the complex numbers* $z \ldots$

A.1 2001 Extension 2 HSC

Question 2

(a) Let $z = 2 + 3i$ and $w = 1 + i$. Find zw and $\dfrac{1}{w}$ in the form $x + iy$. **2**

(b) i. Express $1 + \sqrt{3}i$ in modulus-argument form. **2**

ii. Hence evaluate $\left(1 + \sqrt{3}i\right)^{10}$ in the form $x + iy$. **2**

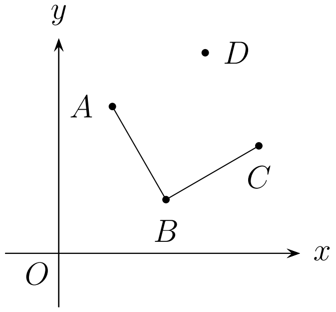
(c) Sketch the region in the complex plane where the inequalities **3**

$$|z + 1 - 2i| \leq 3 \quad \text{and} \quad -\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{4}$$

both hold.

(d) Find all solutions of the equation $z^4 = -1$. **3**
Give your answers in modulus-argument form.

(e) In the diagram the vertices of a triangle ABC are represented by the complex numbers z_1, z_2 and z_3, respectively. The triangle is isosceles and right-angled at B.



i. Explain why $(z_1 - z_2)^2 = -(z_3 - z_2)^2$. **2**

ii. Suppose D is the point such that $ABCD$ is a square. Find the complex number, expressed in terms of z_1, z_2 and z_3, that represents D. **1**

Question 3

(b) The numbers α, β and γ satisfy the equations

$$\alpha + \beta + \gamma = 3 \qquad \alpha^2 + \beta^2 + \gamma^2 = 1 \qquad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2$$

i. Find the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$ **3**

Explain why α, β and γ are the roots of the cubic equation

$$x^3 - 3x^2 + 4x - 2 = 0$$

ii. Find the values of α, β and γ. **2**

Question 7

(a) Suppose that $z = \dfrac{1}{2}(\cos\theta + i\sin\theta)$ where θ is real.

i. Find $|z|$. **1**

ii. Show that the imaginary part of the geometric series **3**

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

is $\dfrac{2\sin\theta}{5 - 4\cos\theta}$.

iii. Find an expression for **2**

$$1 + \frac{1}{2}\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{2^3}\cos 3\theta + \cdots$$

in terms of $\cos\theta$.

(b) Consider the equation $x^3 - 3x - 1 = 0$.

i. Let $x = \dfrac{p}{q}$ where p and q are integers having no common divisors other than $+1$ and -1. Suppose that x is a root of $ax^3 - 3x + b = 0$, where a and b are integers. **4**

Explain why p divides b and why q divides a. Deduce that $x^3-3x-1 = 0$ does not have a rational root.

ii. Suppose that r, s and d are rational numbers and that $\sqrt{d}$ is irrational. Assume that $r + s\sqrt{d}$ is a root of $x^3 - 3x - 1 = 0$. **4**

Show that $3r^2s + s^3d - 3s = 0$ and show that $r - s\sqrt{d}$ must also be a root of $x^3 - 3x - 1 = 0$.

Deduce from this result and part (i), that no root of $x^3 - 3x - 1 = 0$ can be expressed in the form $r + s\sqrt{d}$ with r, s and d rational.

iii. Show that one root of $x^3 - 3x - 1 = 0$ is $2\cos\dfrac{\pi}{9}$. **1**
You may assume the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

A.2 2002 Extension 2 HSC

Question 2

(a) Let $z = 1 + 2i$ and $w = 1 + i$. Find, in the form $x + iy$,

i. $z\overline{w}$. **1**

ii. $\dfrac{1}{w}$. **1**

(b) On an Argand diagram, shade in the region where the inequalities **3**

$$0 \leq \text{Re}(z) \leq 2 \quad \text{and} \quad |z - 1 + i| \leq 2$$

both hold.

(c) It is given that $2 + i$ is a root of

$$P(z) = z^3 + rz^2 + sz + 20$$

where r and s are real numbers.

i. State why $2 - i$ is also a root of $P(z)$. **1**

ii. Factorise $P(z)$ over the real numbers. **2**

(d) Prove by induction that, for all integers $n \geq 1$, **3**

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

(e) Let $z = 2(\cos\theta + i\sin\theta)$.

i. Find $\overline{1 - z}$. **1**

ii. Show that the real part of $\dfrac{1}{1-z}$ is $\dfrac{1 - 2\cos\theta}{5 - 4\cos\theta}$ **2**

iii. Express the imaginary part of $\dfrac{1}{1-z}$ in terms of θ. **1**

Question 3

(a) The equation $4x^3 - 27x + k = 0$ has a double root. Find the possible values of k. **2**

A.3 2003 Extension 2 HSC

Question 2

(a) Let $z = 2 + i$ and $w = 1 - i$. Find, in the form $x + iy$,

i. $z\overline{w}$. **1**

ii. $\dfrac{4}{z}$. **1**

(b) Let $\alpha = -1 + i$.

i. Express α in modulus-argument form. **2**

ii. Show that α is a root of the equation $z^4 + 4 = 0$. **1**

iii. Hence, or otherwise, find a real quadratic factor of the polynomial z^4+4. **2**

(c) Sketch the region in the complex plane where the inequalities **3**

$$|z - 1 - i| < 2 \quad \text{and} \quad 0 < \arg(z - 1 - i) < \frac{\pi}{4}$$

hold simultaneously.

(d) By applying De Moivre's theorem and by also expanding $(\cos\theta + i\sin\theta)^5$, express $\cos 5\theta$ as a polynomial in $\cos\theta$. **3**

(e) ⚠ Suppose that the complex number z lies on the unit circle, and $0 \le \arg(z) \le \frac{\pi}{2}$. **2**

Prove that $2\arg(z + 1) = \arg(z)$.

A.4 2004 Extension 2 HSC

Question 2

(a) Let $z = 1 + 2i$ and $w = 3 - i$. Find, in the form $x + iy$,

i. zw. **1**

ii. $\overline{\left(\dfrac{10}{z}\right)}$. **1**

(b) Let $\alpha = 1 + i\sqrt{3}$ and $\beta = 1 + i$.

i. Find $\dfrac{\alpha}{\beta}$ in the form $x + iy$. **1**

ii. Express α in modulus-argument form. **2**

iii. Given that β has the modulus-argument form **1**

$$\beta = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

find the modulus-argument form of $\dfrac{\alpha}{\beta}$.

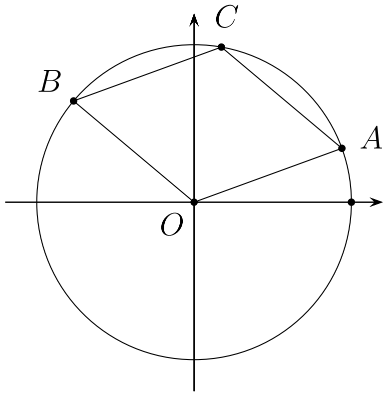
iv. Hence find the exact value of $\sin\dfrac{\pi}{12}$. **1**

(c) Sketch the region in the complex plane where the inequalities **3**

$$|z + \overline{z}| \leq 1 \quad \text{and} \quad |z - i| \leq 1$$

hold simultaneously.

(d) The diagram shows two distinct points A and B that represent the complex numbers z and w respectively. The points A and B lie on the circle of radius r centred at O. The point C representing the complex number $z + w$ also lies on this circle.



i. Using the fact that C lies on the circle, show geometrically that $\angle AOB = \frac{2\pi}{3}$. **2**

ii. Hence show that $z^3 = w^3$. **2**

iii. Show that $z^2 + w^2 + zw = 0$. **1**

Question 4

(a) Let α, β and γ be the zeros of the polynomial $p(x) = 3x^3 + 7x^2 + 11x + 51$.

i. Find $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **1**

ii. Find $\alpha^2 + \beta^2 + \gamma^2$. **2**

iii. Using part (ii), or otherwise, determine how many of the zeros of $p(x)$ are real. Justify your answer. **1**

Question 7

(b) Let α be a real number and suppose that z is a complex number such that

$$z + \frac{1}{z} = 2\cos\alpha$$

i. By reducing the above equation to a quadratic equation in z, solve for z and use De Moivre's theorem to show that **3**

$$z^n + \frac{1}{z^n} = 2\cos n\alpha$$

ii. Let $w = z + \dfrac{1}{z}$. Prove that **2**

$$w^3 + w^2 - 2w - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$$

iii. Hence, or otherwise, find all solutions of **3**

$$\cos\alpha + \cos 2\alpha + \cos 3\alpha = 0$$

in the range $0 \le \alpha \le 2\pi$.

A.5 2005 Extension 2 HSC

Question 2

(a) Let $z = 3 + i$ and $w = 1 - i$. Find, in the form $x + iy$,

i. $2z + iw$. **1**

ii. $\overline{z}w$. **1**

iii. $\dfrac{6}{w}$. **1**

(b) Let $\beta = 1 - i\sqrt{3}$.

i. Express β in modulus-argument form. **2**

ii. Express β^5 in modulus-argument form. **2**

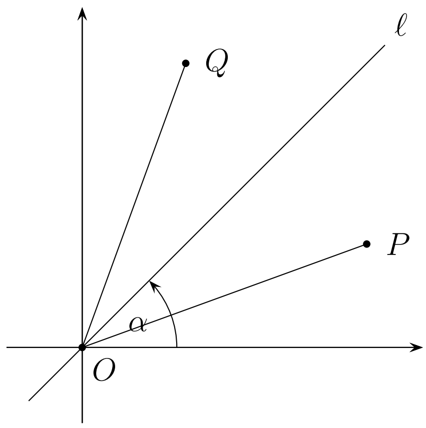
iii. Hence express β^5 in the form $x + iy$. **1**

(c) Sketch the region in the complex plane where the inequalities **3**

$$|z - \overline{z}| < 2 \quad \text{and} \quad |z - 1| \geq 1$$

hold simultaneously.

(d) Let ℓ be the line in the complex plane that passes through the origin and makes an angle α with the positive real axis, where $0 < \alpha < \frac{\pi}{2}$.



The point P represents the complex number z_1, where $0 < \arg(z_1) < \alpha$. The point P is reflected in the line ℓ to produce the point Q, which represents the complex number z_2. Hence $|z_1| = |z_2|$.

i. Explain why $\arg(z_1) + \arg(z_2) = 2\alpha$. **2**

ii. Deduce that $z_1 z_2 = |z_1|^2 (\cos 2\alpha + i \sin 2\alpha)$. **1**

iii. Let $\alpha = \dfrac{\pi}{4}$ and let R be the point that represents the complex number $z_1 z_2$. **1**

Describe the locus of R as z_1 varies.

Question 4

(b) Suppose α, β, γ and δ are the four roots of the polynomial equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

i. Find the values of $\alpha + \beta + \gamma + \delta$ and $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **2**

ii. Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = p^2 - 2q$. **2**

iii. Apply the result in part (ii) to show that $x^4 - 3x^3 + 5x^2 + 7x - 8 = 0$ cannot have four real roots. **1**

iv. By evaluating the polynomial at $x = 0$ and $x = 1$, deduce that the polynomial equation $x^4 - 3x^3 + 5x^2 + 7x - 8 = 0$ has exactly two real roots. **2**

Question 6

(b) Let n be an integer greater than 2. Suppose ω is an n-th root of unity and $\omega \neq 1$.

i. By expanding the left, show that **2**

$$\left(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1}\right)(\omega - 1) = n$$

ii. Using the identity $\dfrac{1}{z^2 - 1} = \dfrac{z^{-1}}{z - z^{-1}}$, or otherwise, prove that **1**

$$\frac{1}{\cos 2\theta + i \sin 2\theta - 1} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta}$$

provided that $\sin\theta \neq 0$.

iii. Hence, if $\omega = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$, find the real part of $\dfrac{1}{\omega - 1}$. **1**

iv. Deduce that $1 + 2\cos\dfrac{2\pi}{5} + 3\cos\dfrac{4\pi}{5} + 4\cos\dfrac{6\pi}{5} + 5\cos\dfrac{8\pi}{5} = -\dfrac{5}{2}$. **1**

v. By expressing the left hand side of the equation in part (iv) in terms of $\cos\dfrac{\pi}{5}$ and $\cos\dfrac{2\pi}{5}$, find the exact value, in surd form, of $\cos\dfrac{\pi}{5}$. **3**

A.6 2006 Extension 2 HSC

Question 2

(a) Let $z = 3 + i$ and $w = 2 - 5i$. Find, in the form $x + iy$,

i. z^2. **1**

ii. $\overline{z}w$. **1**

iii. $\dfrac{w}{z}$. **1**

(b) i. Express $\sqrt{3} - i$ in modulus-argument form. **2**

ii. Express $\left(\sqrt{3} - i\right)^7$ in modulus-argument form. **2**

iii. Hence express $\left(\sqrt{3} - i\right)^7$ in the form $x + iy$. **1**

(c) Find, in modulus-argument form, all solutions of $z^3 = -1$. **2**

Question 3

(c) Two of the zeros of $P(x) = x^4 - 12x^3 + 59x^2 - 138x + 130$ are $a + ib$ and $a + 2ib$, where a and b are real and $b > 0$.

i. Find the values of a and b. **3**

ii. Hence, or otherwise, express $P(x)$ as the product of quadratic factors with real coefficients. **1**

Question 4

(a) The polynomial $p(x) = ax^3 + bx + c$ has a multiple zero at 1 and has a remainder 4 when divided by $x + 1$. Find a, b, c. **3**

A.7 2007 Extension 2 HSC

Question 2

(a) Let $z = 4 + i$ and $w = \overline{z}$. Find, in the form $x + iy$,

i. w. **1**

ii. $w - z$. **1**

iii. $\dfrac{z}{w}$. **1**

(b) i. Write $1 + i$ in the form $r(\cos\theta + i\sin\theta)$. **2**

ii. Hence, or otherwise, find $(1 + i)^{17}$ in the form $a + ib$, where a and b are integers. **3**

(c) The point P on the Argand diagram represents the complex number z, where z satisfies **3**

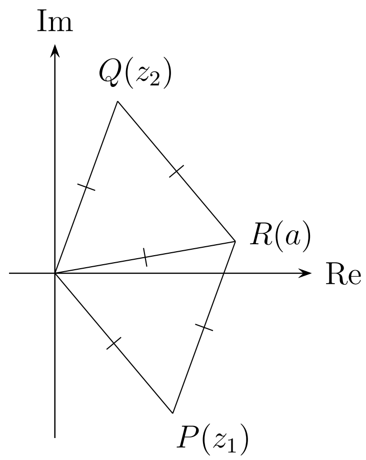
$$\frac{1}{z} + \frac{1}{\overline{z}} = 1$$

Give a geometrical description of the locus of P as z varies.

(d) The points P, Q and R on the Argand diagram represent the complex numbers z_1, z_2 and a respectively.

The triangles OPR and OQR are equilateral with unit sides, so $|z_1| = |z_2| = |a| = 1$.

Let $\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.



i. Explain why $z_2 = \omega a$. **1**

ii. Show that $z_1 z_2 = a^2$. **1**

iii. Show that z_1 and z_2 are the roots of $z^2 - az + a^2 = 0$. **2**

Question 4

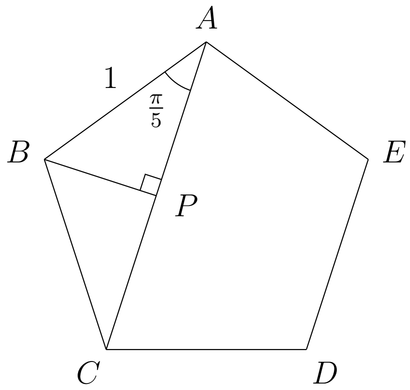
(d) The polynomial $P(x) = x^3 + qx^2 + rx + s$ has real coefficients. It has three distinct zeros, α, $-\alpha$ and β.

i. Prove that $qr = s$. **3**

ii. The polynomial does not have three real zeros. Show that two of the zeros are purely imaginary. (A number is purely imaginary if it is of the form iy, with y real and $y \neq 0$.) **2**

Question 5

(d) In the diagram, $ABCDE$ is a regular pentagon with sides of length 1. The perpendicular to AC through B meets AC at P.



Copy or trace this diagram into your writing booklet.

i. Let $u = \cos \frac{\pi}{5}$. **2**

Use the cosine rule in $\triangle ACD$ to show that $8u^3 - 8u^2 + 1 = 0$.

ii. One root of $8x^3 - 8x^2 + 1 = 0$ is $\frac{1}{2}$. **2**

Find the other roots of $8x^3 - 8x^2 + 1 = 0$ and hence find the exact value of $\cos \frac{\pi}{5}$.

Question 8

(b) i. Let n be a positive integer. Show that if $z^2 \neq 1$, then **2**

$$1 + z^2 + z^4 + \cdots + z^{2n-2} = \left(\frac{z^n - z^{-n}}{z - z^{-1}}\right) z^{n-1}$$

ii. By substituting $z = \cos\theta + i\sin\theta$, where $\sin\theta \neq 0$ in to part (i), show that **3**

$$1 + \cos 2\theta + \cdots + \cos(2n-2)\theta + i\left[\sin 2\theta + \cdots + \sin(2n-2)\theta\right] = \frac{\sin n\theta}{\sin\theta}\left[\cos(n-1)\theta + i\sin(n-1)\theta\right]$$

iii. Suppose $\theta = \dfrac{\pi}{2n}$. Using part (ii), show that **3**

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} = \cot\frac{\pi}{2n}$$

A.8 2008 Extension 2 HSC

Question 2

(a) Find real numbers a and b such that $(1+2i)(1-3i) = a+ib$. **2**

(b) i. Write $\dfrac{1+i\sqrt{3}}{1+i}$ in the form $x+iy$, where x and y are real. **2**

ii. By expression both $1+i\sqrt{3}$ and $1+i$ in modulus-argument form, write $\dfrac{1+i\sqrt{3}}{1+i}$ in modulus-argument form. **3**

iii. Hence find $\cos\dfrac{\pi}{12}$ in surd form. **1**

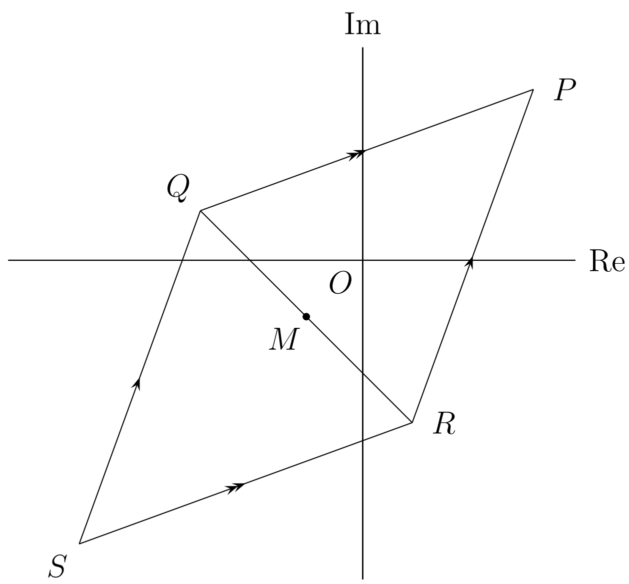
iv. By using the result of part (ii), or otherwise, calculate $\left(\dfrac{1+i\sqrt{3}}{1+i}\right)^{12}$. **1**

(c) The point P on the Argand diagram represents the complex number $z = x+iy$, which satisfies **3**

$$z^2 + \overline{z}^2 = 8$$

Find the equation of the locus of P in terms of x and y. What type of curve is the locus?

(d) The point P on the Argand diagram represents the complex number z. The points Q and R represent the points ωz and $\overline{\omega} z$ respectively, where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. The point M is the midpoint of QR.



i. Find the complex number representing M in terms of z. **2**

ii. The point S is chosen so that $PQSR$ is a parallelogram. **2**

Find the complex number represented by S.

Question 3

(b) Let $p(z) = 1 + z^2 + z^4$.

i. Show that $p(z)$ has no real zeros. **1**

ii. Let α be a zero of $p(z)$.

(α) Show that $\alpha^6 = 1$. **1**

(β) Show that α^2 is also a zero of $p(z)$. **1**

Question 5

(b) Let $p(x) = x^{n+1} - (n+1)x + n$, where n in a positive integer.

i. Show that $p(x)$ has a double zero at $x = 1$. **2**

ii. By considering concavity, or otherwise, show that $p(x) \geq 0$ for $x \geq 0$. **1**

iii. Factorise $p(x)$ when $n = 3$. **2**

Question 6

(a) Let ω be the complex number satisfying $\omega^3 = 1$ and $\text{Im}(\omega) > 0$. The cubic polynomial, $p(z) = z^3 + az^2 + bz + c$, has zeros 1, $-\omega$ and $-\overline{\omega}$. **3**

Find $p(z)$.

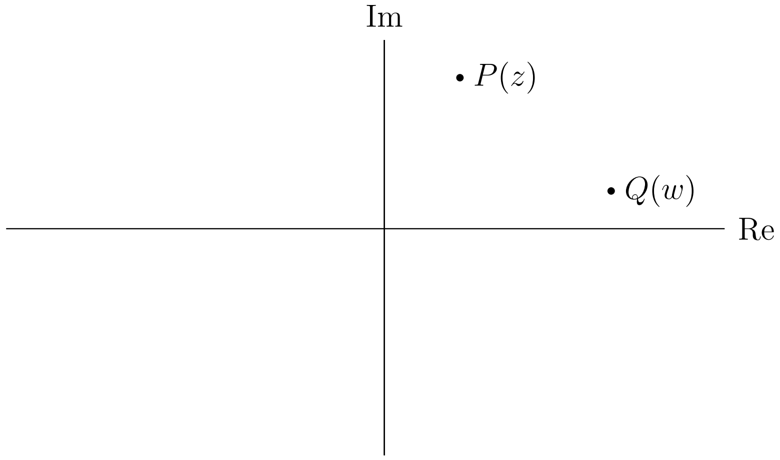
A.9 2009 Extension 2 HSC

Question 2

(a) Write i^9 in the form $a + ib$ where a and b are real. **1**

(b) Write $\dfrac{-2+3i}{2+i}$ in the form $a + ib$ where a and b are real. **1**

(c) The points P and Q on the Argand diagram represent the complex numbers z and w respectively.



Copy the diagram into your writing booklet, and mark on it the following points:

i. the point R representing iz **1**

ii. the point S representing $\overline{w}$ **1**

iii. The point T representing $z + w$. **1**

(d) Sketch the region in the complex plane where the inequalities $|z - 1| \leq 2$ and $-\frac{\pi}{4} \leq \arg(z - 1) \leq \frac{\pi}{4}$ hold simultaneously. **2**

(e) i. Find all the 5th roots of -1 in modulus-argument form. **2**

ii. Sketch the 5th roots of -1 on an Argand diagram. **1**

(f) i. Find the square roots of $3 + 4i$. **3**

ii. Hence, or otherwise, solve the equation **2**

$$z^2 + iz - 1 - i = 0$$

Question 3

(c) Let $P(x) = x^3 + ax^2 + bx + 5$, where a and b are real numbers. **3**

Find the values of a and b given that $(x - 1)^2$ is a factor of $P(x)$.

Question 6

(b) Let $P(x) = x^3 + qx^2 + qx + 1$ where $q \in \mathbb{R}$. One zero of $P(x)$ is -1.

i. Show that if α is a zero of $P(x)$ then $\dfrac{1}{\alpha}$ is a zero of $P(x)$. **1**

ii. Suppose that α is a zero of $P(x)$ and α is not real.

(α) Show that $|\alpha| = 1$. **2**

(β) Show that $\text{Re}(\alpha) = \dfrac{1 - q}{2}$. **2**

A.10 2010 Extension 2 HSC

Question 2

(a) Let $z = 5 - i$.

i. Find z^2 in the form $x + iy$. **1**

ii. Find $z + 2\overline{z}$ in the form $x + iy$. **1**

iii. Find $\dfrac{i}{z}$ in the form $x + iy$. **2**

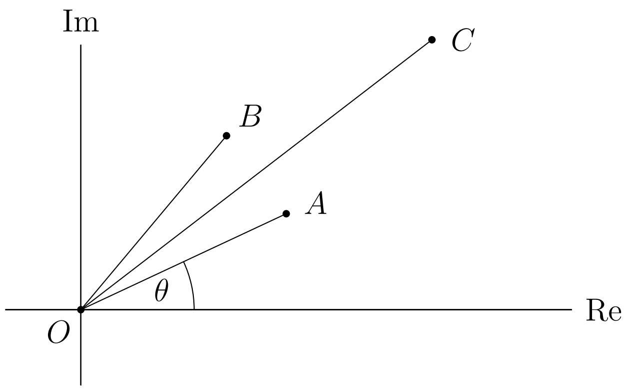
(b) i. Express $-\sqrt{3} - i$ in modulus-argument form. **2**

ii. Show that $\left(-\sqrt{3} - i\right)^6$ is a real number. **2**

(c) Sketch the region in the complex plane where the inequalities $1 \leq |z| \leq 2$ and $0 \leq z + \overline{z} \leq 3$ hold simultaneously. **2**

(d) Let $z = \cos\theta + i\sin\theta$ where $0 < \theta < \dfrac{\pi}{2}$.

On the Argand diagram the point A represents z, the point B represents z^2 and the point C represents $z + z^2$.



Copy or trace the diagram into your writing booklet.

i. Explain why the parallelogram $OACB$ is a rhombus. **1**

ii. Show that $\arg\left(z + z^2\right) = \dfrac{3\theta}{2}$ **1**

iii. Show that $\left|z + z^2\right| = 2\cos\dfrac{\theta}{2}$. **2**

iv. By considering the real part of $z + z^2$, or otherwise, deduce that **1**

$$\cos\theta + \cos 2\theta = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

Question 6

(c) i. Expand $(\cos\theta + i\sin\theta)^5$ using the binomial theorem. **1**

ii. Expand $(\cos\theta + i\sin\theta)^5$ using De Moivre's Theorem, and hence show that **3**

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

iii. Deduce that $x = \sin\left(\frac{\pi}{10}\right)$ is one of the solutions to **1**

$$16x^5 - 20x^3 + 5x - 1 = 0$$

iv. Find the polynomial $p(x)$ such that **1**

$$(x-1)p(x) = 16x^5 - 20x^3 + 5x - 1$$

v. Find the value of a such that $p(x) = (4x^2 + ax - 1)^2$. **1**

vi. Hence find an exact value for $\sin\frac{\pi}{10}$. **1**

Question 7

(b) The graphs of $y = 3x - 1$ and $y = 2^x$ intersect at $(1, 2)$ and at $(3, 8)$. Using these graphs, or otherwise, show that $2^x \geq 3x - 1$ for $x \geq 3$. **1**

(c) Let $P(x) = (n-1)x^n - nx^{n-1} + 1$ where n is an odd integer, $n \geq 3$.

i. Show that $P(x)$ has exactly two stationary points. **1**

ii. Show that $P(x)$ has a double zero at $x = 1$. **1**

iii. Use the graph $y = P(x)$ to explain why $P(x)$ has exactly one real zero other than 1. **2**

iv. Let α be the real zero of $P(x)$ other than 1. **2**
Using part (b) or otherwise, show that $-1 < \alpha \leq -\frac{1}{2}$.

v. Deduce that each of the zeros of $4x^5 - 5x^4 + 1$ has modulus less than or equal to 1. **2**

A.11 2011 Extension 2 HSC

See Examples 17 on page 21, 22 on page 27 and 61 on page 67.

Question 2

(d) i. Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^3$. **1**

ii. Use De Moivre's theorem and your result from part (i) to prove that **3**

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

iii. Hence, or otherwise, find the smallest positive solution of **2**

$$4\cos^3\theta - 3\cos\theta = 1$$

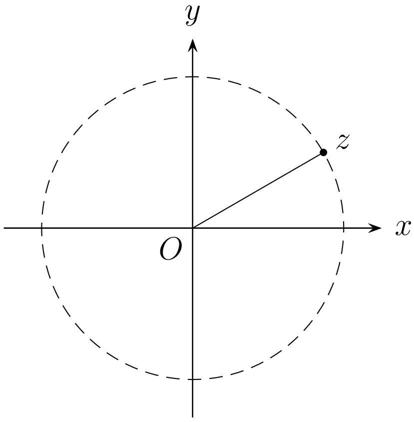
A.12 2012 Extension 2 HSC

1. Let $z = 5 - i$ and $w = 2 + 3i$. **1**

What is the value of $2z + \overline{w}$?

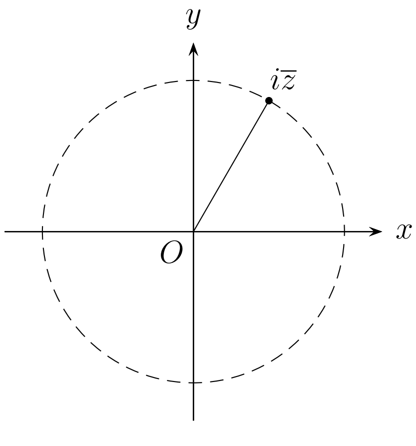
(A) $12 + i$ (B) $12 + 2i$ (C) $12 - 4i$ (D) $12 - 5i$

2. The complex number z is shown on the Argand diagram below. **1**

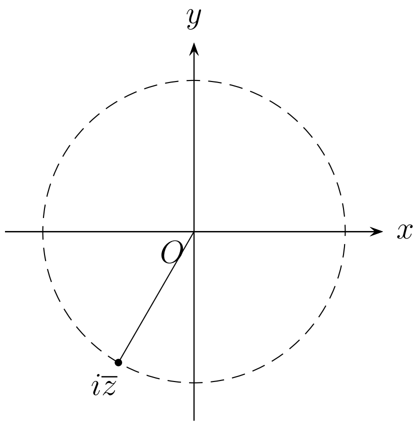


Which of the following best represents $i\overline{z}$?

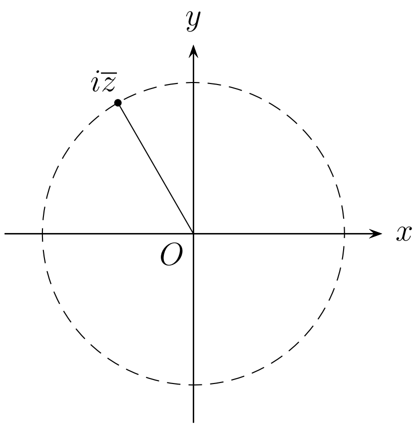
(A)



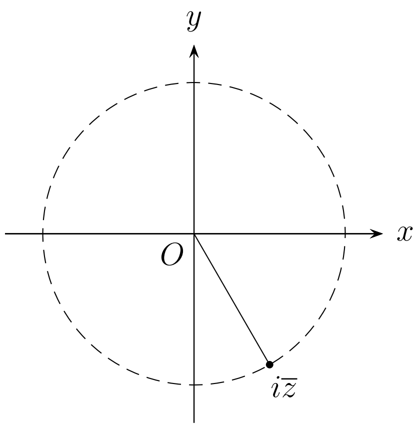
(C)



(B)



(D)

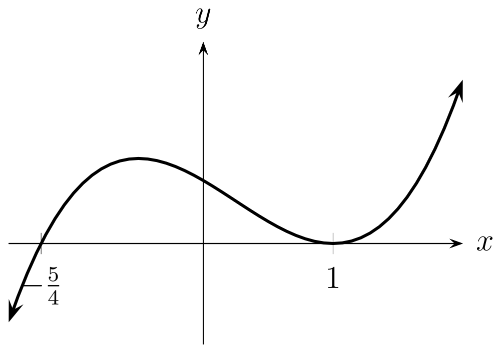


5. The equation $2x^3 - 3x^2 - 5x - 1 = 0$ has roots α, β and γ. **1**

What is the value of $\dfrac{1}{\alpha^3\beta^3\gamma^3}$?

(A) $\frac{1}{8}$ (B) $-\frac{1}{8}$ (C) 8 (D) -8

8. The following diagram shows the graph $y = P'(x)$, the derivative of a polynomial $P(x)$. **1**



Which of the following expressions could be $P(x)$?

(A) $(x-2)(x-1)^3$

(B) $(x+2)(x-1)^3$

(C) $(x-2)(x+1)^3$

(D) $(x+2)(x+1)^3$

Question 11

(a) Express $\dfrac{2\sqrt{5}+i}{\sqrt{5}-1}$ in the form $x+iy$, where x and y are real. **2**

(b) Sketch the region in the complex plane where the inequalities **2**

$$|z+2| \geq 2 \quad \text{and} \quad |z-i| \leq 1$$

both hold.

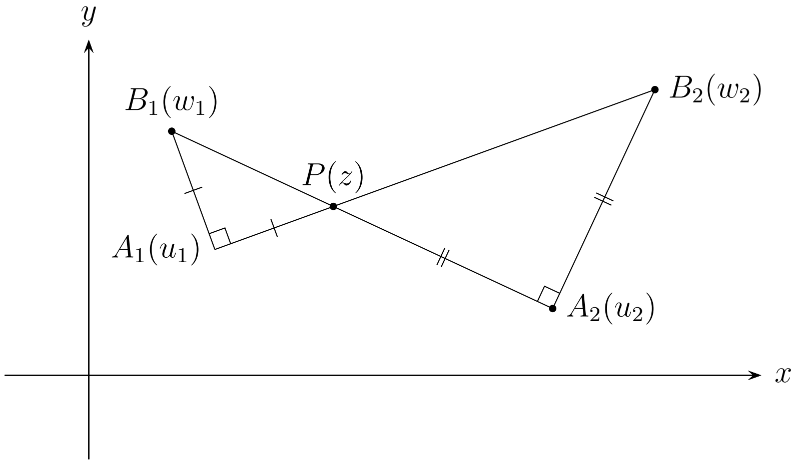
(d) i. Write $z = \sqrt{3} - i$ in modulus-argument form. **2**

ii. Hence express z^9 in the form $x+iy$, where x and y are real. **1**

Question 12

(d) On the Argand diagram the points A_1 and A_2 correspond to the distinct complex numbers u_1 and u_2 respectively. Let P be a point corresponding to a third complex number z.

Points B_1 and B_2 are positioned so that $\triangle A_1PB_1$ and $\triangle A_2B_2P$, labelled in an anti-clockwise direction, are right-angled and isosceles with right angles at A_1 and A_2 respectively. The complex numbers w_1 and w_2 correspond to B_1 and B_2 respectively.



i. Explain why $w_1 = u_1 + i(z - u_1)$. **1**

ii. Find the locus of the midpoint of B_1B_2 as P varies. **2**

Question 15

(b) Let $P(z) = z^4 - 2kz^3 + 2k^2z^2 - 2kz + 1$, where $k \in \mathbb{R}$.

Let $\alpha = x + iy$, where x, $y \in \mathbb{R}$.

Suppose that α and $i\alpha$ are roots of $P(z)$, where $\overline{\alpha} \neq i\alpha$.

i. Explain why $\overline{\alpha}$ and $-i\overline{\alpha}$ are zeros of $P(z)$. **1**

ii. Show that $P(z) = z^2\left(z - k\right)^2 + (kz - 1)^2$. **1**

iii. Hence show that if $P(z)$ has a real zero then **2**

$$P(z) = \left(z^2 + 1\right)\left(z + 1\right)^2 \quad \text{or} \quad P(z) = \left(z^2 + 1\right)\left(z - 1\right)^2$$

iv. Show that all zeros of $P(z)$ have modulus 1. **2**

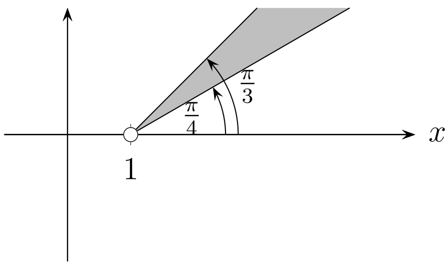
v. Show that $k = x - y$. **1**

vi. Hence show that $-\sqrt{2} \leq k \leq \sqrt{2}$. **2**

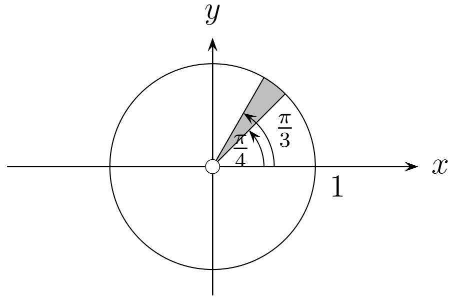
A.13 2013 Extension 2 HSC

5. Which region on the Argand diagram is defined by $\dfrac{\pi}{4} \le |z-1| \le \dfrac{\pi}{3}$?

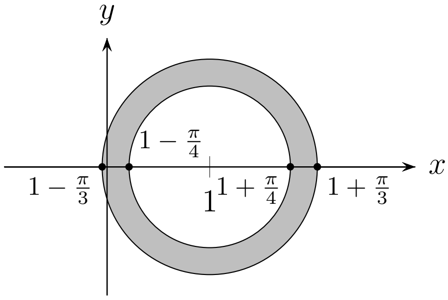
(A)



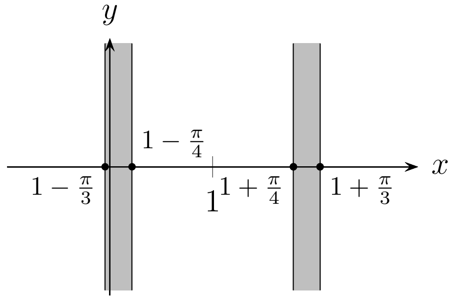
(C)



(B)



(D)



Question 11

(a) Let $z = 2 - i\sqrt{3}$ and $w = 1 = i\sqrt{3}$.

i. Find $z + \overline{w}$. **1**

ii. Express w in modulus-argument form. **2**

iii. Write w^{24} in its simplest form.

(c) Factorise $z^2 + 4iz + 5$. **2**

(e) Sketch the region on the Argand diagram defined by $z^2 + \overline{z}^2 \le 8$. **2**

Question 14

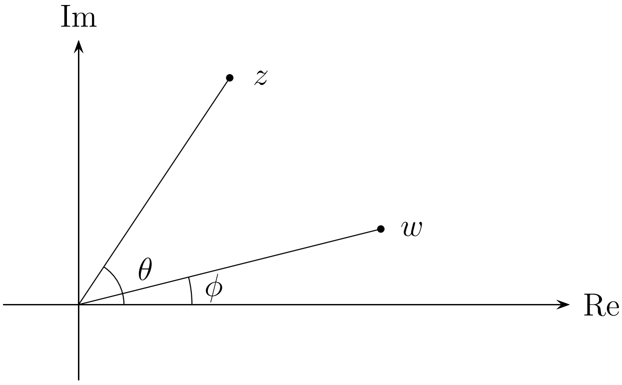
(b) ⚠ Let $z_2 = 1 + i$ and, for $n > 2$, let **3**

$$z_n = z_{n-1}\left(1 + \frac{i}{|z_{n-1}|}\right)$$

Use mathematical induction to prove that $|z_n| = \sqrt{n}$ for all integers $n \ge 2$.

Question 15

(a) The Argand diagram shows complex numbers w and z with arguments ϕ and θ respectively, where $\phi < \theta$. The area of the triangle formed by O, w and z is A. **3**



Show that $z\overline{w} - w\overline{z} = 4iA$

A.14 2014 Extension 2 HSC

2. The polynomial $P(z)$ has real coefficients, and $z = 2 - i$ is a root of $P(z)$. **1**

Which quadratic polynomial must be a factor of $P(z)$.

(A) $z^2 - 4z + 5$ (B) $z^2 + 4z + 5$ (C) $z^2 - 4z + 3$ (D) $z^2 + 4z + 3$

4. Given $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, which expression is equal to $(\overline{z})^{-1}$? **1**

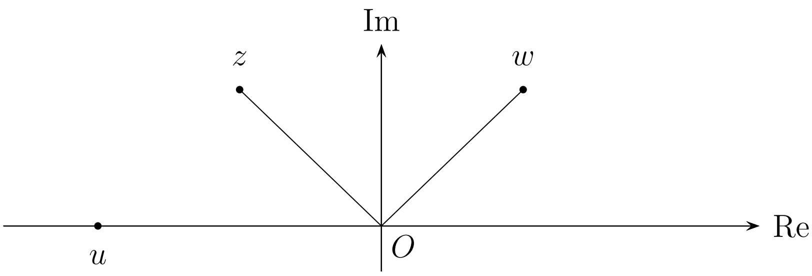
(A) $\frac{1}{2}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

(B) $2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

(C) $\frac{1}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(D) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

8. The Argand diagram shows the complex numbers w, z and u, where w lies in the first quadrant, z lies in the second quadrant and u lies on the negative real axis. **1**



Which statement could be true?

(A) $u = zw$ and $u = z + w$

(B) $u = zw$ and $u = z - w$

(C) $z = uw$ and $u = z + w$

(D) $z = uw$ and $u = z - w$

Question 11

(a) Consider the complex numbers $z = -2 - 2i$ and $w = 3 + i$.

i. Express $z + w$ in modulus-argument form. **2**

ii. Express $\dfrac{z}{w}$ in the form $x + iy$, where x and y are real numbers. **2**

(c) Sketch the region in the Argand diagram where $|z| \le |z - 2|$ and $-\dfrac{\pi}{4} \le \arg z \le \dfrac{\pi}{4}$. **3**

Question 12

(b) It can be shown that $4\cos^3\theta - 3\cos\theta = \cos 3\theta$. (Do NOT prove this.)

Assume that $x = 2\cos\theta$ is a solution of $x^3 - 3x = \sqrt{3}$.

i. Show that $\cos 3\theta = \dfrac{\sqrt{3}}{2}$. **1**

ii. Hence, or otherwise, find the three real solutions of $x^3 - 3x = \sqrt{3}$. **1**

Question 14

(a) Let $P(x) = x^5 - 10x^2 + 15x - 6$.

i. Show that $x = 1$ is a root of $P(x)$ of multiplicity three. **2**

ii. Hence, or otherwise, find the two complex roots of $P(x)$. **2**

A.15 2015 Extension 2 HSC

2. What value of z satisfies $z^2 = 7 - 24i$? **1**

(A) $4 - 3i$

(B) $-4 - 3i$

(C) $3 - 4i$

(D) $-3 - 4i$

5. Given that $z = 1 - i$, which expression is equal to z^3? **1**

(A) $z = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

(B) $z = 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

(C) $z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

(D) $z = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

9. The complex number z satisfies $|z - 1| = 1$. **1**

What is the greatest distance that z can be from the point i on the Argand diagram?

(A) 1 (B) $\sqrt{5}$ (C) $2\sqrt{2}$ (D) $\sqrt{2} + 1$

Question 11

(a) Express $\dfrac{4 + 3i}{2 - i}$ in the form $x + iy$, where x and y are real. **2**

(b) Consider the complex numbers $z = -\sqrt{3} + i$ and $w = 3\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$.

i. Evaluate $|z|$. **1**

ii. Evaluate $\arg(z)$. **1**

iii. Find the argument of $\dfrac{z}{w}$. **1**

Question 12

(a) The complex number z is such that $|z| = 2$ and $\arg(z) = \dfrac{\pi}{4}$.

Plot each of the following complex numbers on the same half-page Argand diagram.

i. z. **1**

ii. $u = z^2$. **1**

iii. $v = z^2 - \overline{z}$. **1**

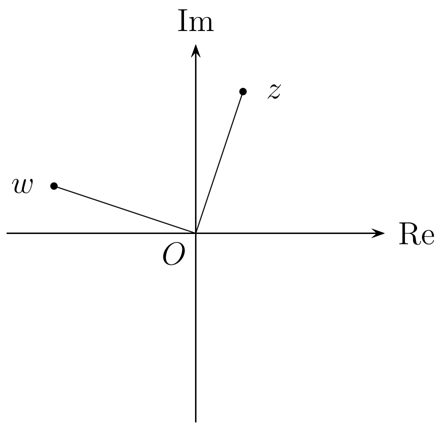
(b) The polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has roots $a + ib$ and $a + 2ib$ where a and b are real and $b \neq 0$.

i. By evaluating a and b, find all the roots of $P(x)$. **3**

ii. Hence or otherwise, find one quadratic polynomial with real coefficients that is a factor of $P(x)$. **1**

A.16 2016 Extension 2 HSC

4. The Argand diagram shows the complex numbers z and w, where z lies in the first quadrant and w lies in the second quadrant. **1**



Which complex number could lie in the 3rd quadrant?

(A) $-w$

(B) $2iz$

(C) $\overline{z}$

(D) $w - z$

5. Multiplying a non-zero complex number by $\dfrac{1-i}{1+i}$ results in a rotation about the origin on an Argand diagram. **1**

What is the rotation?

(A) Clockwise by $\dfrac{\pi}{4}$

(B) Clockwise by $\dfrac{\pi}{2}$

(C) Anticlockwise by $\dfrac{\pi}{4}$

(D) Anticlockwise by $\dfrac{\pi}{2}$

Question 11

(a) Let $z = \sqrt{3} - i$.

i. Express z in modulus-argument form. **2**

ii. Show that z^6 is real. **2**

iii. Find a positive integer n such that z^n is purely imaginary. **1**

Question 12

(a) Let $z = \cos\theta + i\sin\theta$.

i. By considering the real part of z^4, show that $\cos 4\theta$ is **2**

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

ii. Hence, or otherwise, find an expression for $\cos 4\theta$ involving only powers of $\cos\theta$. **1**

Question 13

(d) Suppose $p(x) = ax^3 + bx^2 + cx + d$ with a, b, c and $d \in \mathbb{R}$, $a \neq 0$.

i. Deduce that if $b^2 - 3ac < 0$ then $p(x)$ cuts the x axis only once. **2**

ii. If $b^2 - 3ac = 0$ and $p\left(-\frac{b}{3a}\right) = 0$, what is the multiplicity of the root $x = -\frac{b}{3a}$? **2**

Question 16

(a) i. The complex numbers $z = \cos\theta + i\sin\theta$ and $w = \cos\alpha + i\sin\alpha$, where $-\pi < \theta \leq \pi$ and $-\pi < \alpha \leq \pi$ satisfy **3**

$$1 + z + w = 0$$

By considering the real and imaginary parts of $1 + z + w$, or otherwise, show that 1, z and w form the vertices of an equilateral triangle in the Argand diagram

ii. Hence, or otherwise, show that if the three non-zero complex numbers $2i$, z_1 and z_2 satisfy **2**

$$|2i| = |z_1| = |z_2| \text{ AND } 2i + z_1 + z_2 = 0$$

then they form the vertices of an equilateral triangle in the Argand diagram.

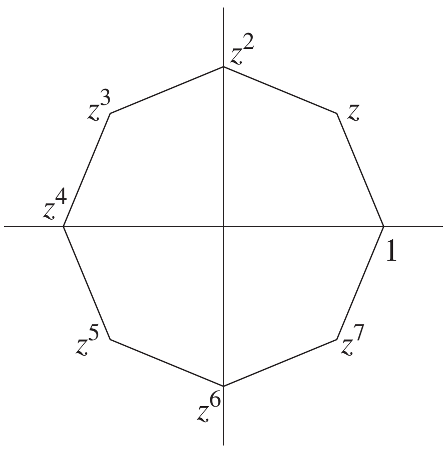
(b) i. The complex numbers 0, u and v form the vertices of an equilateral triangle in the Argand diagram. **2**

Show that $u^2 + v^2 = uv$

ii. Give an example of non-zero complex numbers u and v, so that 0, u and v form the vertices of an equilateral triangle in the Argand diagram. **1**

A.17 2017 Extension 2 HSC

1. The complex number z is chosen so that 1, z, ..., z^7 form vertices of the regular polygon as shown. **1**



Which polynomial equation has all of these complex numbers as roots?

(A) $x^7 - 1 = 0$

(B) $x^7 + 1 = 0$

(C) $x^8 - 1 = 0$

(D) $x^8 + 1 = 0$

3. Which complex number lies in the region $2 < |z-1| < 3$? **1**

(A) $1+\sqrt{3}i$ (B) $1+3i$ (C) $2+i$ (D) $3-i$

6. It is given that $z = 2+i$ is a root of $z^3 + az^2 - 7x + 15 = 0$, where $a \in \mathbb{R}$. **1**

What is the value of a?

(A) -1 (B) 1 (C) 7 (D) -7

Question 11

(a) Let $z = 1 - \sqrt{3}i$ and $w = 1 + 1$.

i. Find the exact value of the argument of z. **1**

ii. Find the exact value of the argument of $\dfrac{z}{w}$. **2**

(c) Sketch the region in the Argand diagram where **2**

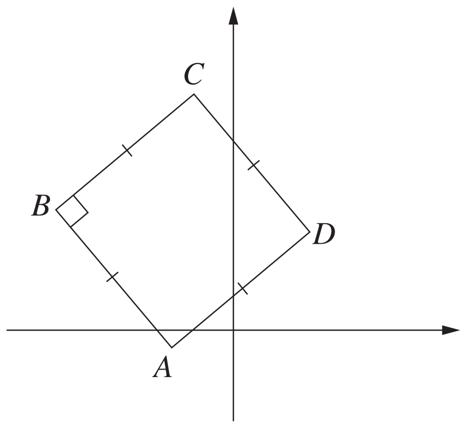
$$-\frac{\pi}{4} \leq \arg z \leq 0 \text{ and } |z - 1 + i| \leq 1$$

Question 12

(b) Solve the quadratic equation $z^2 + (2+3i)z + (1+3i) = 0$, giving your answers in the form $a+bi$, where a and b are real numbers. **3**

Question 13

(e) The points A, B, C and D on the Argand diagram represents the complex numbers a, b, c and d respectively. The points form a square as shown on the diagram. **2**



By using vectors, or otherwise, show that $c = (1+i)d - ia$.

Question 16

(a) Let $\alpha = \cos\theta + i\sin\theta$, where $0 < \theta < 2\pi$.

i. Show that $\alpha^k + \alpha^{-k} = 2\cos k\theta$, for any integer k. **1**

Let $C = \alpha^{-n} + \cdots + \alpha^{-1} + 1 + \alpha + \cdots + \alpha^n$, where n is a positive integer.

ii. By summing the series, prove that **3**

$$C = \frac{\alpha^n + \alpha^{-n} - \left(\alpha^{n+1} + \alpha^{-(n+1)}\right)}{(1-\alpha)(1-\overline{\alpha})}$$

iii. Deduce, from parts (i) and (ii), that **2**

$$1 + 2(\cos\theta + \cos 2\theta + \cdots + \cos n\theta) = \frac{\cos n\theta - \cos(n+1)\theta}{1 - \cos\theta}$$

iv. Show that $\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{n\pi}{n}$ is independent of n. **1**

A.18 2018 Extension 2 HSC

6. Which complex number is a 6th root of i? **1**

(A) $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

(B) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

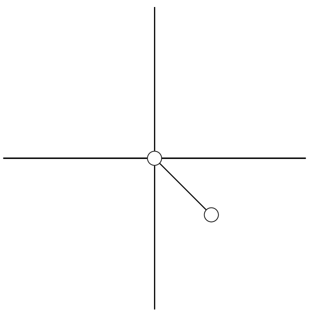
(C) $-\sqrt{2} + \sqrt{2}i$

(D) $-\sqrt{2} - \sqrt{2}i$

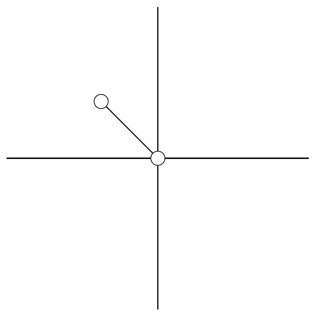
7. Which diagram best represent the solutions to the equation **1**

$$\arg(z) = \arg(z + 1 - i)$$

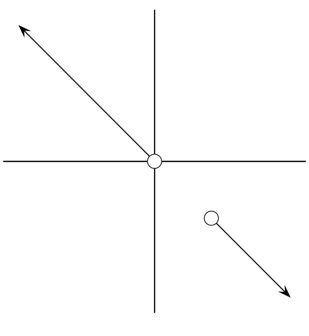
(A)



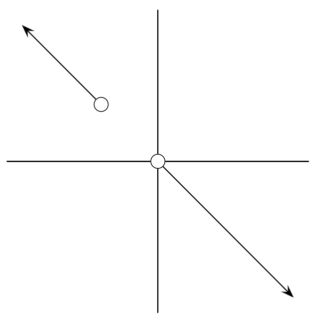
(C)



(B)



(D)



Question 11

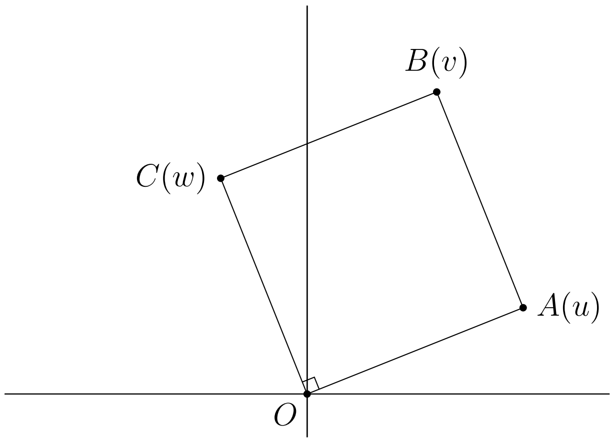
(a) Let $z = 2 + 3i$ and $w = 1 - i$.

i. Find zw. **1**

ii. Express $\overline{z} - \dfrac{2}{w}$ in the form $x + iy$, where x and y are real numbers. **2**

(d) The points A, B and C on the Argand diagram represent the complex numbers u, v and w respectively.

The points O, A, B and C form a square as shown on the diagram.



It is given that $u = 5 + 2i$.

i. Find w. **1**

ii. Find v. **1**

iii. Find $\arg\left(\dfrac{w}{v}\right)$. **1**

Question 13

(b) Let $z = 1 - \cos 2\theta + i \sin 2\theta$, where $0 < \theta \leq \pi$.

i. Show that $|z| = 2\sin\theta$. **2**

ii. Show that $\arg(z) = \dfrac{\pi}{2} - \theta$. **2**

Question 15

(b) i. Use De Moivre's theorem and the expansion of $(\cos\theta + i\sin\theta)^8$ to show that **2**

$$\sin 8\theta = \binom{8}{1}\cos^7\theta\sin\theta - \binom{8}{3}\cos^5\theta\sin^3\theta + \binom{8}{5}\cos^3\theta\sin^5\theta - \binom{8}{7}\cos\theta\sin^7\theta$$

ii. Hence, show that **3**

$$\frac{\sin 8\theta}{\sin 2\theta} = 4\left(1 - 10\sin^2\theta + 24\sin^4\theta - 16\sin^6\theta\right)$$

A.19 2019 Extension 2 HSC

1. What is the value of $(3-2i)^2$? **1**

(A) $5-12i$

(B) $5+12i$

(C) $13-12i$

(D) $13+12i$

8. Let z be a complex number such that $z^2 = -i\overline{z}$. **1**

Which of the following is a possible value for z?

(A) $\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$

(B) $\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$

(C) $\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}i$

(D) $\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i$

Question 11

(a) Let $z = 1+3i$ and $w = 2-i$.

i. Find $z+\overline{w}$. **1**

ii. Express $\dfrac{z}{w}$ in the form $x+iy$, where x and y are real numbers. **2**

(e) Let $z = -1+i\sqrt{3}$.

i. Write z in modulus-argument form. **2**

ii. Find z^3, giving your answer in the form $x+iy$, where x and y are real numbers. **2**

Question 12

(a) Sketch the region defined by $\dfrac{\pi}{4} \le \arg(z) \le \dfrac{\pi}{2}$ and $\mathrm{Im}(z) \le 1$. **2**

Question 16

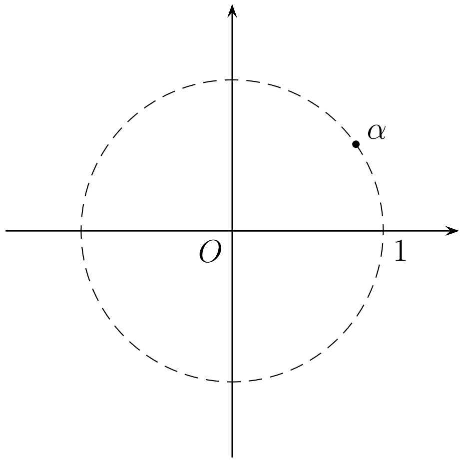
(b) Let $P(z) = z^4 - 2kz^3 + 2k^2z^2 + mz + 1$, where k and m are real numbers.

The roots of $P(z)$ are α, $\overline{\alpha}$, β, $\overline{\beta}$.

It is given that $|\alpha| = 1$ and $|\beta| = 1$.

i. Show that $\Big(\mathrm{Re}(\alpha)\Big)^2 + \Big(\mathrm{Re}(\beta)\Big)^2 = 1$. **2**

ii. The diagram shows the position of α. **2**



Copy or trace the diagram into your writing booklet.

On the diagram, accurately show all possible positions of β.

A.20 2020 Extension 2 HSC

2. Given that $z = 3 + i$ is a root of $z^2 + pz + q = 0$, where p and q are real, what are the values of p and q? **1**

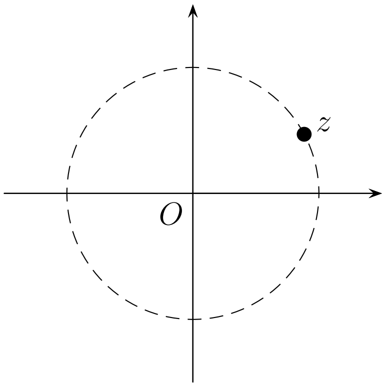
(A) $p = -6,\ q = \sqrt{10}$

(B) $p = -6,\ q = 10$

(C) $p = 6,\ q = \sqrt{10}$

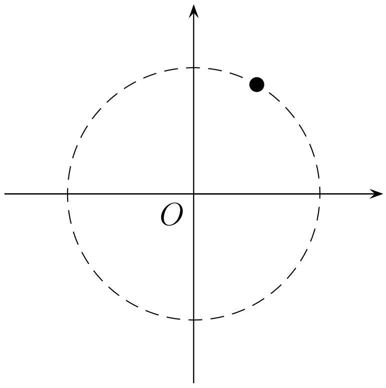
(D) $p = 6,\ q = 10$

4. The diagram shows the complex number z on the Argand diagram. **1**

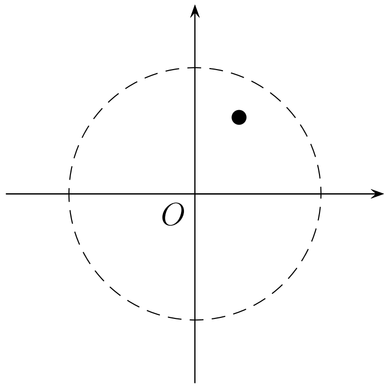


Which of the following diagrams best shows the position of $\dfrac{z^2}{|z|}$?

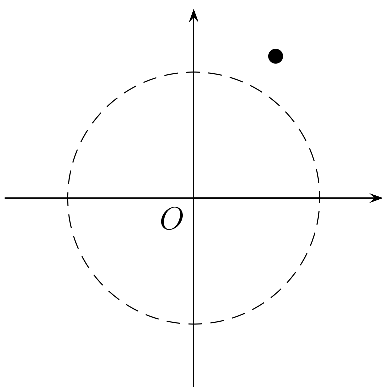
(A)



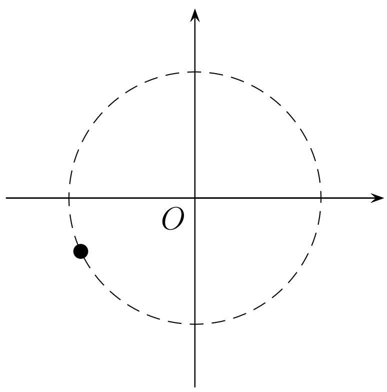
(C)



(B)



(D)



9. What is the maximum value of $\left|e^{i\theta} - 2\right| + \left|e^{i\theta} + 2\right|$ for $0 \le \theta \le 2\pi$? **1**

(A) $\sqrt{5}$ (B) 4 (C) $2\sqrt{5}$ (D) 10

Question 11

(a) Consider the complex numbers $w = -1 + 4i$ and $z = 2 - i$

i. Evaluate $|w|$. **1**

ii. Evaluate $w\overline{z}$. **2**

(e) Solve $z^2 + 3z + (3 - i) = 0$, giving your answer(s) in the form $a + bi$, where a and b are real. **4**

Question 13

(d) i. Show that for any integer n, $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$. **1**

ii. By expanding $\left(e^{in\theta} + e^{-in\theta}\right)^4$, show that **3**

$$\cos^4\theta = \frac{1}{8}\left(\cos(4\theta) + 4\cos(2\theta) + 3\right)$$

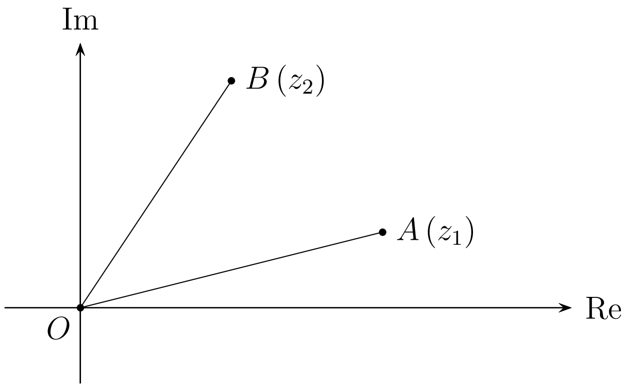
iii. *Note:* For later, after Topic 27 **Further Integration** **2**

Hence, or otherwise, find $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$.

Question 14

(a) Let z_1 be a complex number and $z_2 = e^{\frac{i\pi}{3}} z_1$.

The diagram shows points A and B which represent z_1 and z_2 respectively, in the Argand plane.



i. Explain why triangle OAB is an equilateral triangle. **2**

ii. Prove that $\quad {z_1}^2 + {z_2}^2 = z_1 z_2$. **3**

A.21 2021 Extension 2 HSC

10. Consider the two non-zero complex numbers z and w as vectors. **1**

Which of the following expressions is the projection of z onto w ?

(A) $\dfrac{\text{Re}(zw)}{|w|}w$

(B) $\left|\dfrac{z}{w}\right| w$

(C) $\text{Re}\left(\dfrac{z}{w}\right) w$

(D) $\dfrac{\text{Re}(z)}{|w|}w$

Question 11

(a) The complex numbers $z = 2e^{i\frac{\pi}{2}}$ and $w = 6e^{i\frac{\pi}{6}}$ are given. **2**

Find the value of zw, giving the answer in the form $re^{i\theta}$.

(b) Find $\displaystyle\sum_{n=1}^{5}(i)^n$ **2**

(d) i. Find the two square roots of $-i$, giving the answers in the form $x + iy$, where x and y are real numbers. **2**

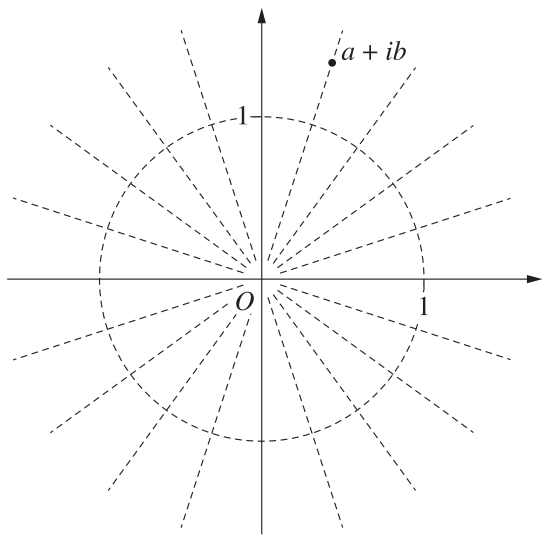
ii. Hence or otherwise, solve $z^2 + 2x + 1 + i = 0$ giving your solutions in the form $a + ib$ where a and b are real numbers. **2**

(e) The complex numbers $z = 5 + i$ and $w = 2 - 4i$ are given. **2**

Find $\dfrac{\overline{z}}{w}$, giving your answer in Cartesian form.

Question 13

(a) The location of the complex number $a + ib$ is shown on the diagram below. On the diagram provided, indicate the locations of all of the fourth roots of the complex number $a + ib$. **2**



Question 14

(c) Using de Moivre's theorem and the binomial expansion of $(\cos\theta + i\sin\theta)^5$, or otherwise, show that **2**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

(d) By using part (i), or otherwise, show that **3**

$$\operatorname{Re}\left(e^{i\frac{\pi}{10}}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}$$

Question 16

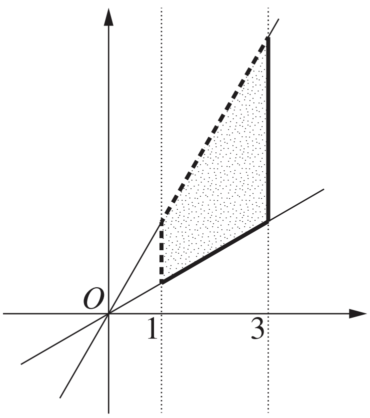
(c) ⚠ Sketch the region of the complex plane defined by $\operatorname{Re}(z) \geq \operatorname{Arg}(z)$ where $\operatorname{Arg}(z)$ is the principal argument of z. **3**

A.22 2022 Extension 2 HSC

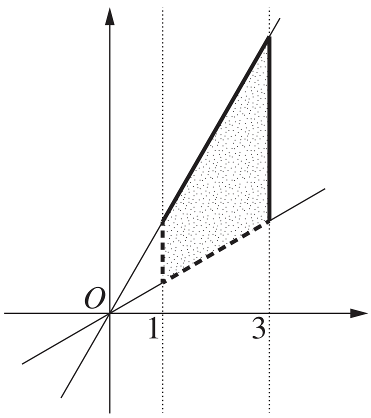
1. Let R be the region in the complex plane defined by $1 < \operatorname{Re}(z) \leq 3$ and $\frac{\pi}{6} \leq \operatorname{Arg}(z) \leq \frac{\pi}{3}$. **1**

Which diagram best represents the region R?

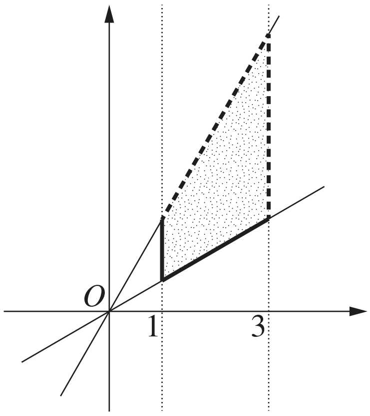
(A)



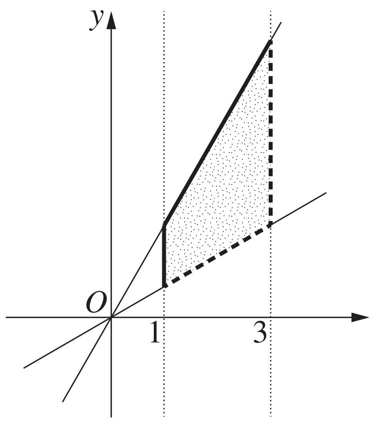
(C)



(B)



(D)



6. It is known that a particular complex number z is NOT a real number. **1**

Which of the following could be true for this number z?

(A) $\overline{z} = iz$

(B) $\overline{z} = |z^2|$

(C) $\text{Re}(iz) = \text{Im}(z)$

(D) $\text{Arg}\left(z^3\right) = \text{Arg}(z)$

Question 11

(a) Express $\dfrac{3-i}{2+i}$ in the form $x+iy$, where x and y are real numbers. **2**

(c) i. Write the complex number $-\sqrt{3}+i$ in exponential form. **2**

ii. Hence, find the exact value of $\left(\sqrt{3}+i\right)^{10}$ giving your answer in the form $x+iy$. **2**

Question 12

(e) Given the complex number $z = e^{i\theta}$, show that $w = \dfrac{z^2-1}{z^2+1}$ is purely imaginary. **3**

Question 13

(c) Consider the equation $z^5+1=0$, where z is a complex number.

i. Sovle the equation $z^5+1=0$ by finding the 5th roots of -1. **2**

ii. Show that if z is a solution of $z^5+1=0$ and $z \neq -1$, then $u = z + \dfrac{1}{z}$ is a solution of $u^2-u-1=0$. **2**

iii. Hence find the exact value of $\cos\dfrac{3\pi}{5}$. **3**

Question 15

(d) The complex number z satisfies $\left|z - \dfrac{4}{z}\right| = 2$. **3**

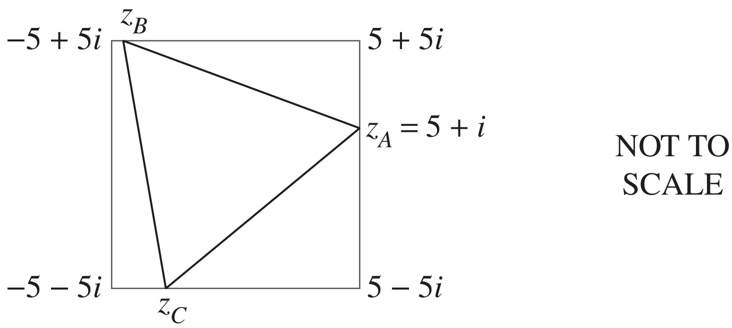
Using the triangle inequality, or otherwise, show that $|z| \leq \sqrt{5}+1$.

Question 16

(a) A square in the Argand plane has vertices **4**

$$5+5i \qquad 5-5i \qquad -5-5i \qquad -5+5i$$

The complex numbers $z_A = 5+i$, z_B and z_C lie on the square and form the vertices of an equilateral triangle, as shown in the diagram.



Find the exact value of the complex number z_B.

(d) Find all the complex numbers z_1, z_2, z_3 that satisfy the following three conditions simultaneously. **3**

$$\begin{cases} |z_1| = |z_2| = |z_3| \\ z_1 + z_2 + z_3 = 1 \\ z_1 z_2 z_3 = 1 \end{cases}$$

A.23 2023 Extension 2 HSC

1. Which of the following is equal to $(a+ib)^3$? **1**

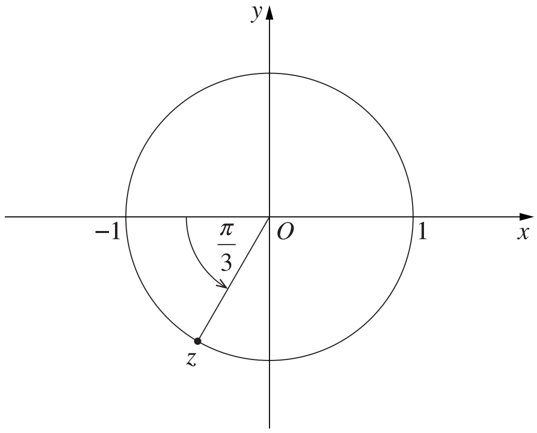
(A) $(a^3 - 3ab^2) + i\,(3a^2b + b^3)$

(B) $(a^3 + 3ab^2) + i\,(3a^2b + b^3)$

(C) $(a^3 - 3ab^2) + i\,(3a^2b - b^3)$

(D) $(a^3 + 3ab^2) + i\,(3a^2b - b^3)$

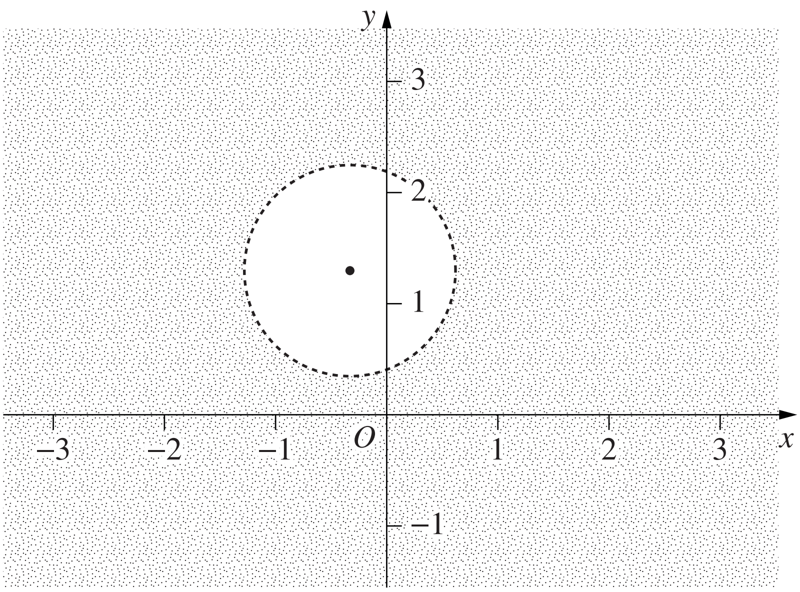
3. A complex number z lies on the unit circle in the complex plane, as shown in the diagram. **1**



Which of the following complex numbers is equal to $\overline{z}$?

(A) $-z$ (B) z^2 (C) $-z^3$ (D) z^4

3. A shaded region on a complex plane is shown. **1**



Which relation best describes the region shaded on the complex plane?

(A) $|z-i| > 2\,|z-1|$ (C) $|z-1| > 2\,|z-i|$

(B) $|z-i| < 2\,|z-1|$ (D) $|z-1| < 2\,|z-i|$

Question 11

(a) Solve the quadratic equation **2**

$$z^2 - 3z + 4 = 0$$

where z is a complex number. Give your answers in Cartesian form.

Question 12

(d) Find the cube roots of $2-2i$. Give your answer in exponential form. **3**

(e) The complex number $2+i$ is a zero of the polynomial

$$P(z)=z^4-3z^3+cz^2+dz-30$$

where c and d are real numbers.

i. Explain why $2-i$ is also a zero of the polynomial $P(z)$. **1**

ii. Find the remaining zeros of the polynomial $P(z)$. **2**

Question 14

(a) Let z be the complex number $z=e^{\frac{i\pi}{6}}$ and w be the complex number $w=e^{\frac{i\pi}{4}}$.

i. By first writing z and w in Cartesian form, or otherwise, show that **3**

$$|z+w|^2=\frac{4-\sqrt{6}+\sqrt{2}}{2}$$

ii. The complex numbers z, w and $z+w$ are represented in the complex plane by the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively, where O is the origin. **2**

Show that $\angle AOC=\dfrac{7\pi}{24}$.

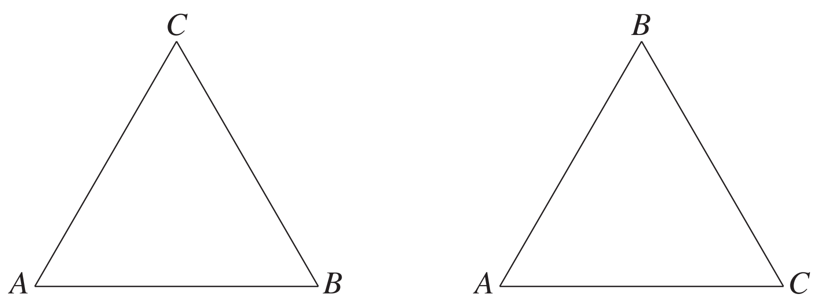
iii. Deduce that $\cos\dfrac{7\pi}{24}=\dfrac{\sqrt{8-2\sqrt{6}+2\sqrt{2}}}{4}$. **1**

Question 16

(a) Let w be the complex number $w=e^{\frac{2i\pi}{3}}$.

i. Show that $1+w+w^2=0$. **2**

The vertices of a triangle can be labelled A, B and C in anticlockwise or clockwise direction, as shown.



ABC is anticlockwise *ABC* is clockwise

Three complex numbers a, b and c are represented in the complex plane by points A, B and C respectively.

ii. Show that if triangle ABC is anticlockwise and equilateral, then **2**

$$a+bw+cw^2=0$$

iii. It can be shown that if triangle ABC is clockwise and equilateral, then $a + bw^2 + cw = 0$. (Do NOT prove this.) **2**

Show that if ABC is an equilateral triangle, then

$$a^2 + b^2 + c^2 = ab + bc + ca$$

(c) ⚠ The complex numbers w and z both have modulus 1, and $\frac{\pi}{2} < \text{Arg}(z) < \pi$, where Arg denotes the principal argument. **3**

For real numbers x and y, consider the complex number $\dfrac{xz + yw}{z}$.

On an xy-plane, clearly sketch the region that contains all points (x, y) for which

$$\frac{\pi}{2} < \text{Arg}\left(\frac{xz + yw}{z}\right) < \pi$$

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

$A = \frac{1}{2}ab\sin C$

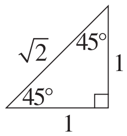
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

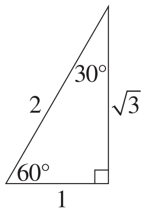
$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \frac{1}{2}r^2\theta$





Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$\cos A = \frac{1-t^2}{1+t^2}$

$\tan A = \frac{2t}{1-t^2}$

$\cos A \cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A \cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A \sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

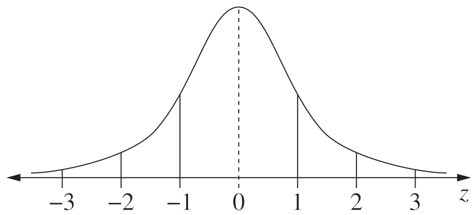
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a) f'(x) a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a) f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^{n} = r^{n}(\cos n\theta + i\sin n\theta)$$
$$= r^{n}e^{in\theta}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^{2}(x-c)$$

References

Arnold, D., & Arnold, G. (2000). *Cambridge Mathematics 4 Unit* (2nd ed.). Cambridge University Press.

Fitzpatrick, J. B. (1991). *New Senior Mathematics – Four Unit Course for Years 12*. Rigby Heinemann.

Lee, T. (2006). *Advanced Mathematics: A complete HSC Mathematics Extension 2 Course* (2nd ed.). Terry Lee Enterprise.

Patel, S. K. (1990). *Excel 4 Unit Maths*. Pascal Press.

Patel, S. K. (2004). *Maths Extension 2* (2nd ed.). Pascal Press.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 12* (1st ed.). Cambridge Education.

Sadler, D., & Ward, D. (2019). *CambridgeMATHS Stage 6 Mathematics Extension 2* (1st ed.). Cambridge Education.